

Flow of Dividends under a Constant Force of Interest

¹Juma Kasozi and ²Jostein Paulsen

¹Department of Mathematics, Makerere University, Box 7062, Kampala, Uganda

²Department of Mathematics, University of Bergen, 5008 Bergen, Norway

Abstract: This study addresses the issue of maximization of dividends of an insurer whose portfolio is exposed to insurance risk. The insurance risk arises from the classical surplus process commonly known as the Cramér-Lundberg model in the insurance literature. To enhance his financial base, the insurer invests in a risk free asset whose price dynamics are governed by a constant force of interest. We derive a linear Volterra integral equation of the second kind and apply an order four Block-by-block method of Paulsen *et al.*^[1] in conjunction with the Simpson rule to solve the Volterra integral equations for each chosen barrier thus generating corresponding dividend value functions. We have obtained the optimal barrier that maximizes the dividends. In the absence of the financial world, the analytical solution has been used to assess the accuracy of our results.

Key words: Risk theory, Volterra equation, block-by-block method, barrier strategy, dividends

INTRODUCTION

The model we consider in this study derives its name from the path breaking work of Lundberg^[2] and Cramér^[3-5]. In this model, at time t , the surplus Y_t of an insurance company evolves according to:

$$Y_t = y + pt - \sum_{i=1}^{N_t} S_i \quad (1)$$

where $y = Y_0 \geq 0$ is the initial reserve, $p > 0$ is the premium rate, $\{N_t\}_{t \in \mathcal{R}^+}$ is a Poisson process with intensity λ , modelling the number of claims (jumps) in $(0, t]$ and $\{S_i\}_{i \in \mathcal{N}}$ is an independent and identically distributed sequence of positive random variables (with distribution F) independent of N , modelling the jump sizes. The distribution F has finite expectation μ and finite variance σ^2 .

The surplus process (1) with the risk averse condition that $p > \lambda E(S_1)$ leads to the unrealistic situation $Y_t \rightarrow \infty$, as $t \rightarrow \infty$. Hence the need to modify the surplus process to take care of this. One way is to predetermine a surplus level b in such a way that when the surplus level hits b , premium income is paid to shareholders as dividends. If the surplus is b and a claim occurs, the claim is paid and no dividends are paid until the surplus hits b again. Eventually and with certainty, ruin will occur at some stage in the future. When ruin occurs, the process stops. The question at hand is 'what b ensures that the insurer pays maximum dividends to shareholders before ruin/bankruptcy?'

We concentrate on barrier strategies in this study although they may not be optimal when compounded by a diffusion in a vibrant financial market in which case other strategies e.g. b and strategies take an upper hand. Under the barrier strategy, we study the expected discounted dividend payments prior to ruin with δ as the discount factor when the model (1) is compounded by a constant force of interest. According to Miller Modigliani theory, this approach can be used as a valuation tool for companies, since the value of a company is exactly expected present value of future dividends^[6]. Their theory was developed under the assumption of perfect capital markets, rational behavior and perfect certainty.

De Finetti^[7] underscored the importance of dividend payments in the economic considerations and management of insurance companies. His work culminated into a barrier strategy as being the optimal strategy. Dividend barrier models have a long history of risk theory^[8-10]. The later work on optimal dividends was extensively discussed by Borch^[11,12]. Gerber^[13,14] studied dividend payouts in (1) with the safety of the company in mind. Most literature is available in Gerber^[15] and the references therein. In many later papers, the problem is formulated and solved in the framework of the Brownian motion model, see e.g. Jeanblanc-Picqué and Shiryaev^[16], Radner and Shepp^[17], Paulsen and Gjessing^[18], Højgaard and Taksar^[19], Taksar and Zhou^[20], Asmussen and Taksar^[21], Asmussen *et al.*^[22], Taksar^[23], Chouilli *et al.*^[24] and Paulsen^[25]. None of the papers in the literature take advantage of the vast knowledge of Volterra integral equations as we do with this work. More to integral equations, we use numerical methods in this study. Before moving a step, we define a dividend and a barrier strategy.

Corresponding Author: Juma Kasozi, ¹Department of Mathematics, Makerere University, P.O. Box 7062, Kampala, Uganda

Definition 1: Dividends are taxable payments declared by the insurer's board of directors and given to shareholders out of the company's current or retained earnings.

Definition 2: A barrier strategy b pays out any surplus above b as dividends immediately so that the surplus is brought back to b . Below b , nothing is paid out.

EXACT SOLUTION TO (1)

To give a mathematical formulation of the models in this study, we take it that all processes and random variables are defined on the stochastic basis $(\Omega, \mathcal{F}, \{f_t\}_{t \in \mathcal{R}^+}, P)$ satisfying the usual conditions, i.e.

f_t is rightly continuous and P -complete. Here, Ω is an abstract sample space whose elements is denoted as ω ; \mathcal{F} is an σ -algebra on Ω ; P is a probability measure and $\{f_t\}_{t \in \mathcal{R}^+}$ is a filtration. A filtration means an increasing and right continuous family of sub σ -algebras of \mathcal{F} , that is, for $0 \leq s \leq t, f_s \subset f_t \subset \mathcal{F}$ and $f_t = f_{t+} = \bigcap_{s>t} f_s$.

Using Itô's formula, the infinitesimal generator for Y is given by the following integral-differential operator:

$$Ag(y) = pg'(y) + \lambda \int_0^\infty (g(y-x) - g(y))dF(x). \tag{2}$$

Now introduce a dividend barrier strategy b so that in the infinitesimal time interval of length dt , the insurer pays the non-negative amount dD_t^b in dividends. With such a strategy, (1) becomes:

$$Y_{t+dt}^b = Y_t^b + pdt - d \sum_{i=1}^{N_t} S_i - dD_t^b, \quad Y_0^b = y > 0.$$

Let:

$$V_b(y) = E^y \left[\int_0^{T_y^b} e^{-\delta t} dD_t^b \right] \tag{3}$$

denote the expected present value of dividend payments prior to ruin, that is, our dividend value function. Here, $\delta > 0$ is the discount factor and time of ruin, $T_y^b = \inf \{t \geq 0 : Y_t^b < 0\}$.

In the said time interval $[t, t + dt]$, a claim occurs with probability $\lambda p dt$. Hence, there will be no claim with probability $1 - \lambda dt$. By law of total probability:

$$V_b(y) = (1 - \delta dt)(1 - \lambda dt)V_b(y + pdt) + \lambda dt \int_0^\infty V_b(y-s)dF(s); \quad 0 < y < b \tag{4}$$

Substituting $V + pdtV'$ for $V_b(y + pdt)$ into (4) and simplifying gives:

$$pV' - (\lambda + \delta)V + \lambda \int_0^\infty V_b(y-s)dF(s) = 0 \tag{5}$$

Specifically, let the jumps be $\text{Exp}(\alpha)$, so that the expectation $\mu = \frac{1}{\alpha}$. Then:

$$pV' - (\lambda + \delta)V + \alpha \lambda \int_0^y V_b(y-s)e^{-\alpha s} ds = 0 \tag{6}$$

Put x for $y - s$ into (6), partially differentiate and eliminate integral term to get:

$$pV'' + (\alpha p - \lambda - \delta)V' - \alpha \delta V = 0; \tag{7}$$

$$V_b(0) = 0, \quad V_b'(b) = 1$$

The solution to the specific case (7) which is also a solution for the general case (5), up to the barrier, is of the form^[25]:

$$V_b(y) = \frac{g(y)}{g'(b)} \tag{8}$$

where, in the case of (7), $g(y) = (\alpha + m_1)e^{m_1 y} - (\alpha + m_2)e^{m_2 y}$. Here, $m_1 < 0$ and $m_2 > 0$ are the roots of the characteristic equation corresponding to (7). Above the barrier, the solution becomes $V_b(y) = V_b(b) + y - b$. Infact $g(y)$ is any solution of $Ag(y) = \delta g(y)$ where A is given by (2). We observe that (5) takes this form.

Remark 1: For a given initial y , let b^* denote the optimal value of b , thmaximizese value that maximises the expected present value of dividends. We see that this b is the value that maximizes (8) thus minimizing $g'(b)$. Hence b^* , is the solution of $g''(b^*) = 0$. The b that maximizes (8) is:

$$b^* = \frac{1}{m_1 - m_2} \log \frac{m_2^2(\alpha + m_2)}{m_1^2(\alpha + m_1)}$$

else $b^* = y$.

All calculations were done on an IBM PC with 1024 MB RAM. The language was FORTRAN and we

used DOUBLE PRECISION to get satisfactory accuracy. Of course slower programs like Splus, Matlab, Maple or Mathematica could have been used but at the expense of a considerably longer computing time. We obtained several valuable functions corresponding to the various chosen barriers. The optimal barrier $b^* = 10.270110$ gives the V^* (maximum dividend value function), all other value functions are less than this V^* despite $b > b^*$. Later, we numerically compute the same dividend value functions using the block-by-block method developed in Paulsen *et al.*^[1] and compare the results. These results and ideas herein will be handy in numerically computing value functions in the Cramér-Lundberg model compounded by a constant force of interest.

NUMERICAL SOLUTION TO (1)

Here, our risk process is still (1) whereby Y is the same process as the one described earlier 2 and $Y^\infty = Y$ thus:

$$Y_t^b = y + pt - \sum_{i=1}^{N_t} S_i - D_t^b, \quad Y_0^b = y > 0. \quad (9)$$

The dividend process D^b is nondecreasing and continuous and $F(0) = 0$. The total expected present value of dividends with discount factor δ and $Y_0^b = y$ is given by (3) where T_y^b is ruin time, equal to infinity if ruin never occurs. Note that when $b < \infty$ then $P(T_y^b < \infty) = 1$ a.s. We use Theorem 1 below to find solutions to (3). Our strategy is to find the associated Volterra integral equation so that we use the methods in Paulsen *et al.*^[1].

Theorem 1: Assume model (9) and let A be given by (2). Let $V_b(y)$ be bounded and twice continuously differentiable on $(0, b)$ with a bounded first derivative there. If $V_b(y)$ solves $AV_b(y) = \delta V_b(y)$ on $0 < y < b$, together with the conditions:

$$V_b(y) = 0 \quad \text{on } y < 0, \quad V_b(0) = 0 \text{ if } \sigma_P^2 > 0, \\ V_b'(b) = 1, \quad V_b(y) = V_b(b) + y - b \quad \text{on } y > b.$$

Then $V_b(y)$ is given by (3).

Proof: Paulsen and Gjessing^[18].

Remark 2: In addition to Remark 1, b^* has geometric characterization. We have the expected present value of dividends:

$$V_b(y) = \begin{cases} V_b(y), & 0 \leq y \leq b \\ V_b(b) + y - b, & y > b. \end{cases}$$

At the junction $y = b$, $V_b(y)$ is continuous and has a continuous first derivative since $V_b'(b) = 1$. Using (8), $V_b''(b) = 0$ is equivalent to $g''(b) = 0$ which in turn means that $b = b^*$ since $g''(b^*) = 0$. This geometric characterization of the optimal value is called the high contact condition in the finance literature and equivalently, the smooth pasting condition in the optimal stopping literature. For details, Dixit and Pindyck^[26].

Theorem 1 together with (2) gives the following version of (5):

$$pV_b'(y) + \lambda \int_0^y (V_b(y-x) - V_b(y)) dF(x) - \delta V_b(y) = 0. \quad (10)$$

Integrating (10) on $[0, z]$ w.r.t. y one concludes that:

$$V_b(y) = \begin{cases} V_b(0) + \int_0^y \left[\frac{\lambda + \delta}{p} - \frac{\lambda}{p} F(y-x) \right] V_b(x) dx, & 0 \leq y \leq b, \\ V_b(b) + y - b, & y > b. \end{cases} \quad (11)$$

For $0 \leq y \leq b$, (11) is a linear Volterra integral equation of the second kind where the kernel is given by

$$K(y, x) = \frac{\lambda}{p} F(y-x) - \frac{\lambda + \delta}{p} \text{ and the forcing function, } \\ h(y) = V_b(0).$$

For any arbitrary starting $g(0)$, $g(y)$ will be the $O(h^4)$ ^[1] numerical solution from the block-by-block method. Then the solution to (11) is given by (8). To find $g'(b)$, we use the approximation $g'(b) \approx \lim_{h \rightarrow 0} \frac{g(b+h) - g(b-h)}{2h}$, where h is the grid size.

We have solved (11) for several values of b . At each run using a FORTRAN program that gives $O(h^4)$ solution to Volterra equations of the second kind (confer Paulsen *et al.*^[1]), we have computed the $g'(b)$. The results indicate that for any two barriers b_1 and b_2 , $0 < b_1 < b_2 < \infty$, $g'(b_1) > g'(b_2)$.

Eventually, some interval $[b_1, b_2]$ is given $g'(b_1) < g'(b_2)$ for the first time. This interval contains the optimal b^* which gives the optimal value function $V_b(b^*)$. When $b = 12$, $g'(12) > g'(10)$ implying that the b^* lies in the interval $[10, 12]$.

To locate the actual b^* , we use our numerical solution of $Ag(y) = \delta g(y)$ to calculate all the derivatives:

$$g'(ih) \approx \lim_{h \rightarrow 0} \frac{g((i+1)h) - g((i-1)h)}{2h} \quad (12)$$

and then locate the minimizing one. Then $b^* = i^* h$ for the minimizing i^* . The corresponding $g'(b^*)$ value is then used to compute the numerical value function which is given by:

$$V_{b^*}(y) = \begin{cases} \frac{g(y)}{g'(b^*)}, & 0 \leq y \leq b^*, \\ \frac{g(b^*)}{g'(b^*)} + y - b^*, & y > b^*. \end{cases}$$

Table 1 shows the results. From the table, $i^* = 1027$ implying that the $b^* = 10.27$.

Finally, here, we compare the exact and numeric solutions. Let $V_{b^*}^N(y)$ be the numerically calculated value function when a numerical value of b^* , is used and $V_{b^*}^A(y)$ be the exact value function when the exact b^* is used. Table 2 shows the absolute percentage relative error:

$$D(y) = \left| 100 \cdot \frac{V_{b^*}^N(y) - V_{b^*}^A(y)}{V_{b^*}^A(y)} \right|$$

It is seen from Table 2 that the results are extremely good. This then gives us confidence to apply the numerical method to the model later.

MODEL (1) COMPOUNDED BY A CONSTANT FORCE OF INTEREST

The Cramér-Lundberg model for the reserve Y_t of an insurance company at time t with a dividend process

D^b is given by equation (9). A natural generalization of this model is to allow reserves to earn interest. If there is a positive constant force of interest r , the reserves now become

$$Y_t^b = y + pt + r \int_0^t Y_s^b ds - \sum_{i=1}^{N_t} S_i - D_t^b, \\ Y_0^b = y > 0$$

Presence of a positive constant force of interest r implies modifying (2) to a new infinitesimal generator for Y now given by the following integro-differential operator:

$$Ag(y) = (ry + p)g'(y) + \lambda \int_0^\infty (g(y-x) - g(y))dF(x) \quad (13)$$

Then Theorem 1 together with (13) gives the integral-differential equation of Volterra type (for $0 \leq y \leq b$):

$$(ry + p)V_b'(y) + \lambda \int_0^\infty (V_b(y-x) - V_b(y))dF(x) - \delta V_b(y) = 0 \quad (14)$$

Table 1: Numerical $V_b(y)$ for Exp(0.5) jumps, $p = 6$, $\lambda = 2$ and $\delta = 0.1$

i	y = ih	$g_b(y)$	$g'(y)$	$V_b(y)$
0	0	5	0	0
1	0.01	5.017489	1.7477949	5.3102001
2	0.02	5.0349559	1.7455936	5.328686
100	1	6.647174	1.5515921	7.034958
200	2	8.1163843	1.3928355	8.5898793
300	3	9.4437971	1.2669783	9.994731
400	4	10.659443	1.1684631	11.281295
500	5	11.788304	1.0927188	12.476012
600	6	12.85121	1.0359868	13.600926
700	7	13.865579	0.9951797	14.674472
800	8	14.846027	0.9677631	15.712117
1000	10	16.752547	0.9451688	17.72986
1024	10.24	16.979344	0.9448811	17.969888
1025	10.25	16.988792	0.9448792	17.979888
1026	10.26	16.998241	0.944878	17.989888
1027	10.27	17.00769	0.9448776	17.999888
1028	10.28	17.017139	0.944878	18.009888
1200	12	18.648711	0.955725	19.736642
1400	14	20.591959	0.9911446	21.793257
1600	16	22.626207	1.0460063	23.946179
1800	18	24.786613	1.1168553	26.232618
2000	20	27.102867	1.2015941	28.683999

Table 2: Comparison of numeric and exact solutions: Exp(0.5) jumps, $\lambda = 2$, $p = 6$, $\delta = 0.1$

y	$V_b^N(y)$	$V_b^A(y)$	D(y)
2	8.5898793	8.5898805	1.444E-05
4	11.281295	11.281297	1.427E-05
6	13.600926	13.600928	1.426E-05
8	15.712117	15.71212	1.426E-05
10	17.72986	17.729863	1.421E-05
12	19.729888	19.736645	1.419E-05

Theorem 2: The integral-differential equation (14) can be represented as the linear Volterra integral equation of the second kind:

$$V_b(y) + \int_0^y K(y,x)V_b(x)dx = h(y),$$

where the kernel:

$$K(y,x) = -\frac{(r + \lambda + \delta) - \lambda F(y-x)}{ry + p},$$

and the forcing function:

$$h(y) = \frac{pV_b(0)}{ry + p}.$$

Proof: Integrating (14) on $[0, z]$ w.r.t. y and letting $v = y - x$ one obtains:

$$(rz + p)V_b(z) - pV_b(0) + \int_0^z (\lambda_p F_p(z-v) - (r + \lambda_p + \delta)) V_b(v)dv = 0 \quad (15)$$

Replacing z with y and v with x in (15) completes the proof.

We know from Theorem 1 that when $y > b$, $V_b(y) = V_b(b) + y - b$. Hence, our value function takes the form:

$$V_b(y) = \begin{cases} \frac{pV_b(0)}{ry + p} + \int_0^y \left(\frac{(r + \lambda_p + \delta) - (\lambda_p F_p(y-x))}{ry + p} \right) V_b(x)dx, & 0 \leq y \leq b, \\ V_b(b) + y - b, & y > b. \end{cases} \quad (16)$$

We carry with us the analysis, results and discussion from the previous section onto (16). The values of $g'(b)$ decrease for $b = 10, 12, 14$. When $b = 16$, $g'(16) > g'(14)$ implying that the b^* lies in the interval $[14, 16]$.

To locate actual b^* , we again calculate all the derivatives using (12) minimizing locate the minimising one. The results indicate that $i^* = 1469$ thus giving $b^* = 14.69$. Figure 1 is a graph of dividend value functions for some select barriers. The optimal dividend value function is the one corresponding to b^* .

When we expect heavy claims, then adequate probability should be assigned to the higher jumps by the distribution function F . Some of the distributions in this class are Lognormal, Pareto and Mixture of exponentials. These distributions are said to have heavy tails.

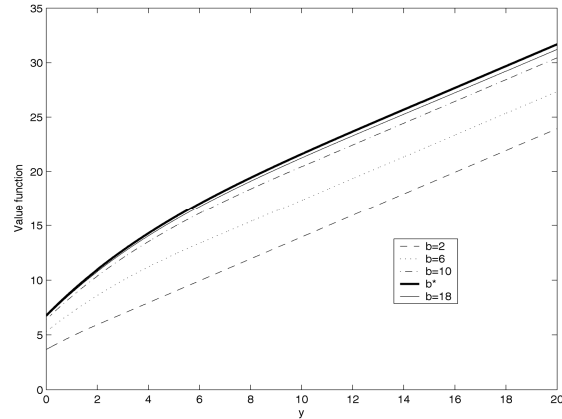


Fig. 1: Numeric value functions for export(0.5) jumps, $\lambda = 2, p = 6, r = 0.05, \delta = 0.1$

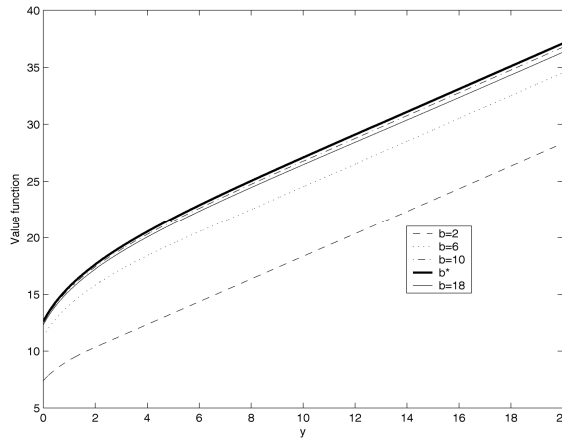


Fig. 2: Numeric value functions for a par (1.5) jumps, $\lambda = 2, p = 6, r = 0.05, \delta = 0.1$

We now report the results when the distribution for S (generic for the S_i) is Pareto. The density function is given by:

$$f(x) = (\alpha - 1)^\alpha \alpha (\alpha - 1 + x)^{-(1+\alpha)}, \quad x > 0, \alpha > 1.$$

Furthermore with this distribution:

$$\bar{F}(x) = \left(\frac{\alpha - 1}{\alpha - 1 + x} \right)^\alpha$$

and

$$E(X) = \int_0^\infty \bar{F}(x)dx.$$

If we set $E(S) = 2$, then $\alpha = 1.5$. To locate actual b^* , we again calculate all the derivatives using (12) minimizing locate the minimising one. The results indicate that $i^* = 1279$ thus giving $b^* = 12.79$. Figure 2 is a graph of dividend value functions for some select barriers. The optimal dividend value function is the one corresponding to b^* .

CONCLUSION

We numerically obtained maximal dividend function in the Cramér-Lundberg model compounded by a constant force of interest using the block-by-block method and a barrier strategy. The results in Table 2 indicate its quality. An extension of this problem is to allow a diffusion in (1) thus:

$$Y_t = pt + \sigma_P W_{P,t} - \sum_{i=1}^{N_t} S_i; \quad t \geq 0.$$

With another diffusion, the return on investment process will no longer be $R_t = rt$ but:

$$R_t = rt + \sigma_R W_{R,t}; \quad t \geq 0.$$

For a detailed description of such a model, see e.g. Paulsen *et al.* [1]. However, when there is a diffusion, band strategies might take an upper hand over barrier strategies and the mathematics changes accordingly.

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REFERENCES

1. Paulsen, J., J. Kasozi and A. Steigen, 2005. A numerical method to find the probability of ultimate ruin in the classical risk model with stochastic return on investments. *Insurance: Maths. and Econ.*, 36: 399-420.
2. Lundberg, F., 1903. I.Approximerad Framställning av Sannolikhetsfunktionen. II. Aterförsäkering av Kollektivrisker. Almqvist and Wiksell, Uppsala.
3. Cramér, H., 1930. On the Mathematical Theory of Risk. Skandia Jubilee Volume, Stockholm.
4. Cramér, H., 1954. On some questions connected with mathematical risk. *Univ. California Publ. Statist.*, 2: 99-125.
5. Cramér, H., 1955. Collective Risk Theory. Skandia Jubilee Volume, Stockholm.
6. Miller, M.H. and F. Modigliani, 1961. Dividend policy, growth and the valuation of shares. *J. Business*, 34: 411-433.
7. De Finetti, B., 1957. Su un'impostazione alternativa della teoria collettiva del rischio. *Trans. 15th Intl. Cong. of Actuaries*, New York, 2: 433-443.
8. Seal, H.L., 1968. *Stochastic Theory of a Risk Business*. New York: Wiley.

9. Bühlmann, H., 1970. *Mathematical methods in risk theory*. Berlin Heidelberg New York: Springer-Verlag.
10. Gerber, H.U., 1973. Martingales in risk theory. *Mitteilungen der Vereinigung Schweizer Versicherungsmatematiker*, 73: 205-216.
11. Borch, K., 1974. *The Mathematical Theory of Insurance*. Lexington, MA: Lexington Books.
12. Borch, K., 1990. *Economics of Insurance*. Amsterdam: North-Holland.
13. Gerber, H.U., 1974. The dilemma between dividends and safety and a generalization of the Lundberg-Cramér formulas. *Scand. Actuarial J.*, pp: 46-57.
14. Gerber, H.U., 1981. On the probability of ruin in the presence of a linear dividend barrier. *Scand. Actuarial J.*, pp: 105-115.
15. Gerber, H.U., 1979. *An Introduction to Mathematical Risk Theory*. S.S. Huebner Foundation Monograph Series No. 8. Homewood, IL: Irwin.
16. Jeanblanc-Picqué, M. and A.N. Shiryaev, 1995. Optimization of the flow of dividends. *Russian Math. Surveys*, 50: 257-277.
17. Radner, R. and L. Shepp, 1996. Risk vs. Profit potential: A model for corporate strategy. *J. Econ. Dynam. Control*, 20: 1373-1393.
18. Paulsen, J. and H.K. Gjessing, 1997. Optimal choice of dividend barriers for a risk process with stochastic return on investments. *Insurance: Math and Econ.*, 20: 215-223.
19. Højgaard, B. and M. Taksar, 1999. Controlling risk exposure and dividend payout schemes: Insurance company example. *Math. Finance*, 9: 153-182.
20. Taksar, M. and X.Y. Zhou, 1998. Optimize risk and dividend control of a company with a debt liability. *Insurance: Math and Econ.*, 22: 105-122.
21. Asmussen, S. and M. Taksar, 1997. Controlled diffusion models for dividend payouts. *Insurance: Maths and Econ.*, 20: 1-15.
22. Asmussen, S., B. Højgaard and M. Taksar, 2000. Optimal risk control and dividend distribution policies. Example of excess-of-loss reinsurance for an insurance corporation. *Finance Stoch.*, 4: 299-324.
23. Taksar, M., 2000. Optimal risk/dividend distribution models. Applications for insurance. *Math. Methods Oper. Res.*, 1: 1-42.
24. Chouilli, T., M. Taksar and X.Y. Zhou, 2002. A diffusion model for optimal dividend distribution for a company with constraints on risk control. To appear *Siam. J. Control Opt.*
25. Paulsen, J., 2003. Optimal dividend payouts for diffusions with solvency constraints. *Finance Stochast.*, 7: 457-473.
26. Dixit, A.K. and R.S. Pindyck, 1994. *Investment under Uncertainty*. First Edn. Princeton University Press.