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On Permutable Subgroups of n-ary Groups

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Abstract: It is proved that every permutable subgroup of a finite n-ary group is subnormal.

Key words: finite n-ary group, permutable n-ary group, subnormal n-ary group

INTRODUCTION

We remind that, the system G = < X, ( )> with one n-ary operation ( ) is called n-ary group[1,2], if it is associative and every one of the equations.

$(a_1\ a_2 \ldots a_{i-1}\ x\ a_{i+1} \ldots a_n) = a$

is solvable in X, where $a_1, \ldots a_n$, a ∈ X, i = 1, 2, …, n. Throughout this study all n-ary groups are finite. Let G be n-ary group and let H is a subgroup of G, then H is called permutable n-ary group if HT=TH for all subgroups T of G.

It is known however, that every permutable subgroup of a finite group is subnormal[3,4]. In this study we prove this property for n-ary groups.

PRELIMINARIES

Notation is standard[2]

X_m^k -the sequence $X_m X_{m+1} \ldots X_k$ (if m = k then $X_m^m = x_m$).

Definition 1: Let G be n-ary group, then $x_1^{k(n-1)}$ is an identity if $\left(x\, x_1^{k(n-1)}\right) = \left(x_1^{k(n-1)} x\right) = x$ for all x ∈ G.

Definition 2: Let G be n-ary group and let x ∈ G, then the sequence of elements $\bar{x}$ of G is called an inverse of x if $x\bar{x}$ is an identity.

Let H ≤ G and x_1^i, y_1^j are sequences of elements of G, where i + j = k(n-1) [k ∈ N], then the symbol $\left[X_1^i H y_1^j\right]$ denote all elements $\left(x_1^i h\, y_1^j\right)$ where h ∈ H.

By analogous of binary groups n-ary subgroup H of a group G is called normal if for any x ∈ G and for any sequence $\bar{x}$ we have $xH\bar{x} = H$.

Definition 3: N-ary subgroup H of a group G is called subnormal in G if:

$H = N_0 \le N_i \le \ldots \le N_{t-1} \le N_t = G$

Where N_i is a normal in N_{i+1}, i = 0, 1, .., t-1.

If H and T are subgroups of n-ary group G, then $\left[\overset{i}{H}\ \overset{n-i}{T}\right]$ is the set of all products $\left(h_1 \ldots h_i\, t_1\, t_{n-i}\right)$, where h_i ∈ H and t_j ∈ T.

Lemma 1[2]: Let H and T are subgroups of n-ary group G such that

$\left[H\ \overset{n-1}{T}\right] = \left[\overset{n-1}{T}\ H\right]$, then B = $\left[H\ \overset{n-1}{T}\right]$ is a subgroup of G and $B \supseteq H$.

Subgroup H of n-ary group G is called permutable if for any subgroup T from G we have

$T \cap H \ne \Phi$ and $\left[H\ \overset{n-1}{T}\right] = \left[\overset{n-1}{T}\ H\right]$

Lemma 2[2]: Let H and T are subgroups of n-ary group G. If H ∩ T ≠ ϕ, then

$\left|\left[H\ \overset{n-1}{T}\right]\right| = \dfrac{|H||T|}{|H \cap T|}$

MAIN RESULTS

We are now to prove the following.

Lemma 3: If H u T are permutable subgroups of n-ary group G, then $\left[H\ \overset{n-1}{T}\right]$ is permutable subgroup of G.

Proof: Let D any subgroup of n-ary group G, then by the definition of permutable subgroup H ∩ D ≠ ϕ. By lemma 1 H≤ $\left[H\ \overset{n-1}{T}\right]$ and it is mean that H∩D ⊆ $\left[H\ \overset{n-1}{T}\right]$ ∩ D ≠ ϕ.

Now since

$$\left[\left[H\ \overset{n-1}{T}\right]\overset{n-1}{D}\right] = \left[H\ \overset{n-2}{T}\ T\, D^{n-1}\right] = \left[H\ \overset{n-2}{T}\ \overset{n-1}{D}\ T\right] =$$

$$\ldots\left[H\ \overset{n-1}{D}\ \overset{n-1}{T}\right] = \left[\overset{n-1}{D}\ H\ \overset{n-1}{T}\right] = \left[\overset{n-1}{D}\left[H\ \overset{n-1}{T}\right]\right]$$

So $\left[H\ \overset{n-1}{T}\right]$ is permutable subgroup in G.

Lemma 4: Let H be a subgroup of n-ary group G. Then if for some element x ∈ G and for some sequence of inverse ($\bar{x}$) of x we have $\left[H\, H_1^{n-1}\right]$ = G, where $H_1 = xH\bar{x}$, then H = H_1.

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Proof: Let x = (a b_1 … b_{n-1}) where a ∈ H and b_i ∈ H_i. Let $\overline{b_i}$ be a sequence of elements from H_1 which are inverses for b_i, i = 1, 2, …, n-1. Then
a = (ab_1 … b_{n-1} b_{n-1} … b) = (x $\overline{b}_{n-1}$ ....$\overline{b}_1$).

It is clear, that b_1 .. b_{n-1} x is the sequence of inverses for a. That means if $\overline{a}$ is any sequence of elements of H that inverse for a , then H = $\left[aH\overline{a}\right]$= $\left[\left(x\overline{b}_{n-1}...b_1\right)H\left(b_1....b_{n-1}\,\overline{x}\right)\right]=\left[xH\overline{x}\right]=H_1$.

Lemma 5: Let x be an element of n-ary group G and let φ_x : G → G a map defined by $\varphi_x(g)=xg\overline{x}$ where g ∈ G and $\overline{x}$ is some sequence that is inverse for x. Then φ_x is an automorphism of G.

Proof: For any sequence of element g_1^n from G we have

$$\varphi_x\left(g_1....g_n\right)=x\left(g_1....g_n\right)\overline{x}=$$
$$\left(x\left(g_1\overline{x}x\right)\left(g_2\overline{x}x\right)....\left(g_{n-1}\overline{x}x\right)g_n\overline{x}\right)=$$
$$\left(\left(xg_1\overline{x}\right)\right)\left(xg_2\overline{x}\right).....\left(xg_n\overline{x}\right)=\left(g_1^{\varphi x}g_2^{\varphi x}....g_n^{\varphi x}\right)$$

So φ_x is an endomorphism of n-ary group G.

If g ∈ G, then φ_x ($\overline{x}gx$) = (x($\overline{x}gx$) $\overline{x}$) = g. It means that ℓ x is an epimorphism. It is obvious that φ_x is an injection.

Theorem: If H is a subgroup of n-ary group G that is permutable with any subgroup of G, then H is a subnormal in G.

Proof: We prove by induction on the order of n-ary group G. let N is the greatest permutable subgroup of G (N≠ G) that contains the subgroup H.

We show that N is a normal subgroup of G. let N is not normal subgroup. By the definition of normal subgroup we can find some x∈ G such that xN$\overline{x}$ ≠ N where **Error! Bookmark not defined.** is some sequence that is inverse of x. let φ_x :G → G defined by φ_x(g) = xg$\overline{x}$ for all g ∈ G. by lemma 5, φ_x-is an automorphism n-ary group G. That means xN$\overline{x}$ is a permutable subgroup of n-ary group G u $\left|N\right|=\left|xN\overline{x}\right|$.

Applying lemma 1 we have $D=\left[\overset{n-1}{N}\,N\right]=\left[N\,\overset{n-1}{N}\right]$ which contains N subgroup n-ary group G, where $N_1=xN\overline{x}$. According to lemma 2 the order of this subgroup is:

$$d=\left|\overset{n-1}{N_1}\,N\right|=\frac{\left|N_1\right|\left|N\right|}{\left|N_1\cap N\right|}$$

Since N ≠ $xN\overline{x}$ *and* $\left|N\right|=\left|xN\overline{x}\right|$, then d > N. But by Lemma 3 subgroup D is permutable in G. That means D = G and this contradict lemma 4. So N is a normal subgroup of g. Since $\left|N\right|<\left|G\right|$ and H is permutable subgroup of N, then by, choosing group G we can conclude that H is subnormal subgroup in N. It means H is a subnormal subgroup of G.

REFERENCES

1. Kurash, A.E., 1974. General Algebra (Lectures 1969-70). Nauka, Moscow.
2. Rusakov, C.A., 1992. Algebraic n-ary systems, MN. Nauka and Technika.
3. Stonehewer, S.E., 1972. Permutable Subgroup of Finite Groups. Math Z., 125: 1-16.
4. Schmidt, R., 1994. Subgroup Lattices of Groups. Walter de Gruyter, Berlin-New York.