American Journal of Applied Sciences 2 (3): 672-675, 2005
ISSN 1546-9239

On Fractional Hold Devices Versus Positive Realness of Discrete Transfer Functions

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Abstract: The appropriate use of fractional order holds $(\beta-\mathrm{FROH})$ of correcting gains $\beta\in[-1,1]$ as an alternative to the classical zero and first order -holds (ZOH, FOH) is discussed related to the positive realness of the associate discrete transfer functions obtained from a given continuous transfer function. It is proved that the minimum direct input-output gain (i.e. the quotient of the leading coefficients of the numerator and denominator of the transfer function) needed for discrete positive realness may be reduced by the choice of β compared to that required for discretization via ZOH.

Key words: Zero, First and Fractional order -holds, Positive realness

INTRODUCTION

β-Fractional Order-Holds and Introductory Background on Positive Realness: The realizable continuous transfer function p(s) = q(s)/n(s) = p´(s) + d, of numerator and denominator polynomials q(s) and n(s) with p´(s) being strictly proper, is positive real $(p\in\{PR\})$ if p(s)$\in \mathbf{R}$ (the set of real numbers), $\forall s\in\mathbf{R}$ and Re (p(s)) ≥ 0 for σ =Re s $\geq 0, \forall s\in\mathbf{C}$ [1, 2] (the set of complex numbers). A necessary condition for a realizable continuous transfer function to be positive real is that it is stable with zero or unity relative degree and with eventually critically stable poles being simple with nonnegative residuals. Positive realness also implies stability of zeros [2-4] and it is a key feature in achieving asymptotic hyperstability via feedback for all nonlinear/time-varying device satisfiying a Popov´s -type inequality [5]. The scalar d is the direct input/output gain, with d=0 if and only if p(s) = p´(s) is strictly proper. Consider the class of β -FROH [including ZOH (β =0) and FOH (β =1)] of transfer function $h_\beta(s)$ leading to the β-dependent discrete transfer functions:

$$g_\beta(z)=Z\lfloor h_\beta(s)p(s)\rfloor=g'_\beta(z)+d_\beta;$$
$$h_\beta(s)=\left[1-\beta+\beta\frac{1+sT}{T}h_0(s)\right]h_0(s) \qquad (1)$$

where Z [.] stands for the z-transform. The transfer function $h_\beta(s)$ is obtained directly [3-4] since the output of the hold device being injected as input to the continuous transfer function is:

$$u(t)=u_k+\frac{\beta}{T}(u_k-u_{k-1})(t-kT) \qquad (2)$$

$\forall t\in[kT,(k+1)T)$ with $u_k=u(kT)$ any sample-indicator integer $k\geq 0$ with T being the sampling period. Note that $h_\beta(s)$ may be directly synthesized with two ZOH´s and a simple linear network. It has been proved [2] that $g_0(z)$ is discrete positive real $(g_0\in\{PR_d\})$ if $p(s)$ is stable (or, in particular, positive real) and biproper (i.e. of zero relative degree) with and a sufficiently large associated direct input-output gain $d_0=d\geq d_{min}>0$. This implies that if $d=0$ (i.e. p(s)=p´(s) is strictly proper) then $g_0\notin\{PR_d\}$ even if $p\in\{PR\}$ with unity relative degree. Positive realness under discretization via $\beta-\mathrm{FROH}$ is now discussed by first defining positive realness with prescribed margins.

Definition: It is said that $g_\beta\in\{PR_d(\varepsilon)\}$, some $\varepsilon\geq 0$, if $\mathrm{Re}\,g_\beta(z)\geq\varepsilon$, $\forall z\in UC:=\{z\in\mathbf{C}:|z|=1\}$. Note that $\{PR_d(0)\}\equiv\{PR_d\}$, $g_\beta\in\{PR_d(\varepsilon)\}$ $\Rightarrow$ $g_\beta\in\{PR_d(\varepsilon')\}$, $\forall\varepsilon'\in[0,\varepsilon)$ and $g_\beta\in\{PR_d(\varepsilon)\}$ for some real $\varepsilon>0$ $\Rightarrow$ $g_\beta\in\{SPR_d\}$; i.e. $g_\beta(z)$ is strictly positive real (since $\underset{z\in UC}{\mathrm{Min}}(\mathrm{Re}\,g_\beta(z))>0$).

Positive Realness of $g_\beta(z)$: Direct simple calculations allow rewriting the first equation in (1) as:

$$g_\beta(z) = (1-\beta z^{-1})(g_0'(z)+d_\beta) + \beta T^{-1} z^{-1}(z-1) g_{01}(z) \tag{3}$$

if $p(s) = p'(s) + d_\beta$ with $g_{01}(z) = (1-z^{-1}) Z(s^{-2} p(s))$ what implies $g_0(z) = (1-\beta z^{-1})(g_0'(z)+d_0)$. Simple calculations with (3) lead to

$$g_\beta(z) = (1+\beta\,\tilde{g}(z))(g_0'(z)+d_\beta); \;\; \tilde{g}(z) = \frac{1}{z}\left[\frac{1}{T}\frac{q_{01}(z)}{q_0(z)} - 1\right] \tag{4}$$

since $g_{01}(z)/g_0(z) = q_{01}(z)/((z-1)\,q_0(z))$ with $q_{01}(z)$ and $q_0(z)$ being the respective numerator polynomials of $g_{01}(z)$ and $g_0(z)$ since their respective denominator polynomials $n_{01}(z)$ and $n_0(z)$ satisfy the constraint $n_{01}(z) = (z-1)\,n_0(z)$ from direct calculations involving z-transforms. Since $p'(s)$ is strictly proper then $g_\beta'(z) = Z[h_\beta(s)p'(s)]$ is strictly proper of unity relative degree and order deg (n(s) if β=0 and (1+ deg (n(s)) if $\beta \neq 0$. Let real constants m_i, $m_s \geq m_i$; $\tilde{m}_i$, $\tilde{m}_s \geq \tilde{m}_i$ be such that:

$$\text{Re}\,\tilde{g}(z) \in [\tilde{m}_i, \tilde{m}_s]; \quad \text{Re}\left(\tilde{g}(z)\,g_0'(z)\right) \in [m_i, m_s]; \; \forall z \in UC \tag{5}$$

Direct calculations using the worst lower-bound minimum bound for $\text{Re}\,(g_\beta(z))$ from (4) via (5) lead to

$$\begin{aligned} &\text{Re}\,g_{\beta(\geq 0)}(z) \geq \varepsilon_0 + \Delta d_\beta + \beta\left[m_i + \left(d_0' + \varepsilon_0 + \Delta d_\beta\right)\tilde{m}_i\right]; \\ &\text{Re}\,g_{\beta(<0)}(z) \geq \varepsilon_0 + \Delta d_\beta - |\beta|\left[m_s + \left(d_0' + \varepsilon_0 + \Delta d_\beta\right)\tilde{m}_s\right] \end{aligned} \tag{6}$$

which hold, respectively, for $\beta \geq 0$ and for $\beta < 0$. The technical subsequent assumption is then used.

Assumption 1: $g_0 \in \{PR_d(\varepsilon)\}$ $(\Rightarrow g_0 \in \{PR_d\})$ and $d_\beta \geq (d_o - \varepsilon_0)$ for some real $\varepsilon_0 \geq 0$, all $\beta - FROH$.

Now, define auxiliary real constants $\overline{m}_\ell$ from m_ℓ, $\tilde{m}_\ell$ from (5) as $\overline{m}_\ell := m_\ell + (d_0' + \varepsilon_0)\tilde{m}_\ell$ $(\ell = i, s)$. From Assumption 1 and the constraints (6), the following result holds:

Theorem 1(Discrete Positive Realness Via Design of β): If Assumption 1 holds then $g_\beta \in \{PR_d(\varepsilon)\}$ with $\text{Re}\,g_\beta(z) \geq \varepsilon$ for some sufficiently small $\varepsilon \geq 0$ and some $\beta - FROH$, $\beta \in [\beta_{min}, \beta_{max}] \subseteq [-1, 1]$ if some of the subsequent items holds:

(i) $$\left|\frac{\varepsilon - \varepsilon_0 - \Delta d_\beta}{\overline{m}_i}\right| \leq \beta \leq 1 \tag{10}$$

provided that

$$\varepsilon - \varepsilon_0 \geq \Delta d_\beta \geq \text{Max}\left(-\varepsilon_0, -\frac{\overline{m}_i}{\tilde{m}_i}\right) \text{ if } \tilde{m}_i \neq 0 \text{ or } \varepsilon - \varepsilon_0 \geq \Delta d_\beta \geq -\varepsilon_0 \text{ if } \vec{m}_i > 0 \text{ and } \tilde{m}_i \neq 0 \tag{11}$$

(ii) $$0 \leq \beta \leq \text{Max}\left(\left|\frac{\varepsilon - \varepsilon_0 - \Delta d_\beta}{\overline{m}_i}\right|, 1\right) \tag{12}$$

provided that

$$-\frac{\overline{m}_i}{\tilde{m}_i} > \Delta d_\beta \ge \varepsilon - \varepsilon_0 \text{ if } \tilde{m}_i \ne 0 \text{ or } \Delta d_\beta \ge \varepsilon - \varepsilon_0 \text{ if } \overline{m}_i < 0 \text{ and } \tilde{m}_i = 0 \quad (13)$$

$$\text{(iii) } \beta < 0 \text{ , } |\beta| \le \text{Max}\left(\left| \frac{\varepsilon + \Delta d_\beta - \varepsilon_0}{\overline{m}_s} \right|, 1 \right) \quad (14)$$

provided that

$$\Delta d_\beta \ge \text{Max}\left(\varepsilon - \varepsilon_{0,} - \frac{\overline{m}_s}{\tilde{m}_s} \right) \text{ if } \tilde{m}_s \ne 0 \text{ or } \Delta d_\beta \ge \varepsilon - \varepsilon_0 \text{ if } \overline{m}_s > 0 \text{ and } \tilde{m}_s \ne 0 \quad (15)$$

$$\text{(iv) } \beta < 0 \text{ , } 1 \ge |\beta| \ge \left| \frac{\varepsilon_0 + \Delta d_\beta - \varepsilon}{\overline{m}_s} \right| \quad (16)$$

provided that

$$\varepsilon - \varepsilon_0 \ge \Delta d_\beta \ge \text{Max}\left(-\varepsilon_{0,} - \frac{\overline{m}_s}{\tilde{m}_s} \right) \text{ if } \tilde{m}_s \ne 0 \text{ or } \varepsilon - \varepsilon_0 \ge \Delta d_\beta \ge -\varepsilon_0 \text{ if } \overline{m}_s < 0 \text{ and } \tilde{m}_s \ne 0 \quad (17)$$

By using (6) with $\varepsilon = \varepsilon_0$, the following result stands:

Theorem 2 (Positive Realness Via β-Froh by Increasing/Decreasing Direct Input/Output Gains): If $g_0 \in \{PR_d(\varepsilon_0)\}$ with $d_0 = d_0' + \varepsilon_0$ then $g_\beta \in \{PR_d(\varepsilon_0)\}$ if $d_\beta = d_0 + \Delta d_\beta$ with $\Delta d_\beta \ge \text{Max}\left(-\varepsilon_0, -\frac{\beta \overline{m}_i}{1 + \beta \tilde{m}_i} \right)$ if $\beta \in [0, 1]$ with $\beta \ne -\frac{1}{\tilde{m}_i}$; and $\Delta d_\beta \ge \text{Max}\left(-\varepsilon_0, \frac{|\beta| \overline{m}_s}{1 - |\beta| \tilde{m}_s} \right)$ if $\beta \in [-1, 0]$ with $|\beta| \ne \frac{1}{\tilde{m}_s}$

Remark 1: Note that the margin of positive realness, compared to that achieved with a ZOH, is improved with smaller positive values $0 < d_\beta < d_0$, since for positive realness of discrete transfer functions the relative degree is required to be zero, the direct input-output gain from Theorem 2 if β< 0 satisfying $|\beta| < \text{Min}\left(1, \frac{1}{|\tilde{m}_s|}\right)$ provided that $\text{Min}(\varepsilon_0, \tilde{m}_s) > 0$. This also holds if $1 \ge \beta > \frac{1}{|\tilde{m}_i|}$ with $\tilde{m}_i < 0$, $|\tilde{m}_i| > 1$ and $\overline{m}_i < 0$, if or if $0 < \beta \le \text{Min}\left(\frac{1}{|\tilde{m}_i|}, 1 \right)$ if $|\tilde{m}_i| < 1$. If the usual constraint $\beta \in [-1, 1]$ is removed then several alternative solutions with $|\beta| > 1$ are useful for such a purpose of achieving positive realness for $0 < d_\beta < d_0$.

Example: Note that Theorems 1-2 are based on obtaining worst–case positive lower-bounds of the $\text{Re}(g_\beta(z))$ where each β-dependent right-hand-side term in (6) is minimized. However, it is possible to obtain refinements from positive lower-bounds via numerical evaluation of the relation:

$$d(\beta) > d_{\min}(\beta) := -\left| \underset{z \in UC}{\text{Min}}\, g_\beta'(z) \right| = -\left| \underset{z \in UC}{\text{Min}}\, Z\left[h_\beta(s) p'(s) \right] \right| \ge 0$$

Proceed in that way with $p'(s) = \frac{1}{s+1} \in \{PR\}$.

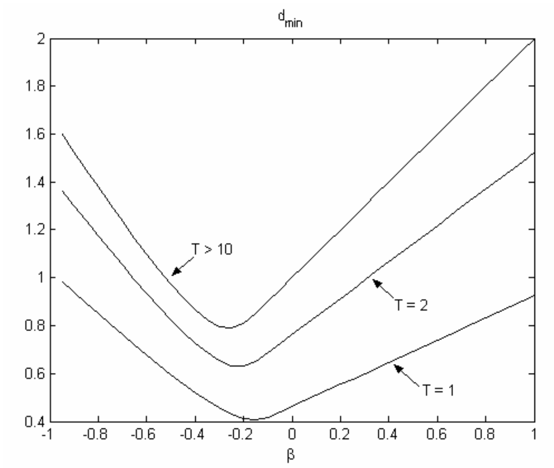

Fig.1: Threshold $d_{min}(\beta)$ to be Used in the Continuous Transfer Function

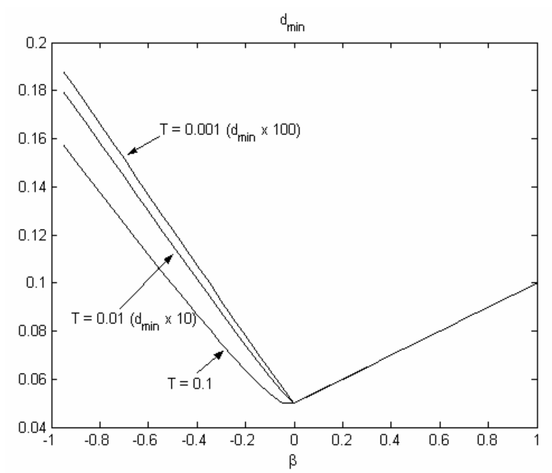

Fig.2: Threshold $d_{min}(\beta)$ to be Used in the Continuous Transfer Function

Fig. 1and 2 display the threshold $d_{min}(\beta)$ to be used in the continuous transfer function to achieve positive realness with a β- FROH for six distinct values of the sampling period ranging from 0.001 to 10 secs. Note that the smaller values of such a threshold are highly dependent on the sampling period and achieved for a range of negative values of β which improve the threshold $d_{min}(0)$ required for β =0.

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