An Overview of Modified Shanks' Conjecture and Comments on its Validity

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Abstract: This study reports the validity of the modified Shanks' conjecture on the planar least squares inverse (PLSI) method of stabilizing two-dimensional (2-D) recursive digital filters. A theoretical procedure proposed based on the Lagrange multiplier method of mathematical optimization. The results indicate that the modified Shanks' conjecture reported by Jury was valid for special classes of 2-D polynomials.

Key words: PLSI, autocorrelation, filter stability, Lagrange multiplier

INTRODUCTION

Two-dimensional (2-D) recursive digital filters find many applications, such as biomedical electronics, image processing, seismic record processing, etc. For a given response characteristic, recursive filters have less hardware requirements and thus, wherever linear phase is not mandatory, recursive filters are preferred. However, the problem of stability is associated with the design of recursive digital filters[1]. The stability tests for 2-D recursive digital filters continue to gain attention from researchers because of their importance in many applications.

It is known in the literature that the least squares inverse (LSI) of a one-dimensional (1-D) polynomial, which represents the denominator of a discrete system function is always stable. This fact has been used for the design of 1-D recursive digital filters with guaranteed stability[2,3]. Shanks' *et al.*[4] proposed the extension of this stabilization technique to 2-D. The 1-D LSI when extended to 2-D is known as planar least squares inverse (PLSI) and Shanks' conjecture has been verified for many practical examples. However, the Shanks conjecture has been shown to be invalid in general. Counterexamples have first been produced and thereafter, a rather simple algebraic procedure has been given to generate polynomials yielding unstable PLSI polynomials of degree (1, 1)[5,6]. Subsequently the modified form of Shanks' conjecture was reported by Jury[7]. According to this modified conjecture, "if the original 2-D polynomial and the corresponding PLSI are of the same degree, then the reciprocal of the PLSI (i.e. the double PLSI) is a stable filter". This conjecture has been verified rigorously for low degree polynomials. However, it has not been proved in general. Jury's conjecture had been verified for a large number of numerical examples. Later, Kayran and King[2] proposed a counterexample to disprove Jury's conjecture. This counterexample shows that Jury's conjecture is not valid in general. In all the above cases, only 2-D quarter-plane (QP) polynomials were considered. In this study, in order to show the validity of the modified form of Shanks' conjecture, 2-D nonsymmetric half-plane (NSHP) polynomials have been taken. It is shown that if the original 2-D NSHP polynomial and its inverse (i.e. PLSI) are of the same degree, then the reciprocal of PLSI (i.e. double PLSI) is a stable filter. In other words, the 2-D PLSI polynomial of an unstable 2-D NSHP polynomial will be definitely stable, provided they both have the same degree.

MATERIALS AND METHODS

Given a transfer function of certain recursive systems,

$$H(z) = \frac{1}{A(z)} \tag{1}$$

A 1-D polynomial $A(z)$ is stable if

$$A(z) \neq 0, \; |z| \leq 1 \tag{2}$$

This is a well known theorem with the fact that the z-transform is defined with positive powers of z. The condition given in (2) states that a 1-D polynomial is stable if and only if all its zeros lie outside the unit

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circle. It is marginally stable if some zero(s) lie on the unit circle[8].

A theoretical procedure is now proposed here for testing the 1-D polynomial $A(z)$ for stability based on the Lagrange multiplier method of mathematical optimization. The Lagrange multiplier method aims to maximize the constant term of the 1-D polynomial and a decision regarding its stability can be made depending upon whether the constant term of the 1-D polynomial is maximizable or not.

Let the given 1-D polynomial $A(z) = \sum_{k=0}^{N} a_k z^k$ be of degree N. It has $(N+1)$ autocorrelation functions γ_s's as given below:

$$\sum_{r=0}^{N} a_r a_{r+s} = \gamma_s, \ s = 0, 1, 2, \cdots, N \qquad (3)$$

Including the given 1-D polynomial $A(z)$, there are totally 2^N number of 1-D polynomials (in general) which have the same autocorrelation coefficients γ_s's as that of $A(z)$ [10,11]. Out of these 2^N number of 1-D polynomials, which are said to form a 'family', only one polynomial is found to be stable. This stable polynomial is one whose constant term a_0 has the highest magnitude. Therefore, given a 1-D polynomial $A(z) = \sum_{k=0}^{N} a_k z^k$, to test whether it is stable or not, we use the Lagrange Multiplier method[9].

Let the stable version of $A(z)$ be $A'(z)$. In this method one has to maximize the function $f = a'_0$ satisfying the constraints g_i, given as

$$g_i = \sum_{r=0}^{N} a'_r a'_{r+s} - \gamma_s = 0, \ s = 0, 1, 2, \cdots, N \qquad (4)$$

where,

$$\gamma_s = \sum_{r=0}^{N} a_r a_{r+s}, \ s = 0, 1, 2, \cdots, N \qquad (5)$$

that is

$$g_i = 0, \ i = 0, 1, 2, \cdots, N \qquad (6)$$

For clarity, we briefly discuss the method as follows. Let us form the Lagrange Function $L(a'_0, \lambda_j)$ such that

$$L(a'_0, \lambda_j) = f + \sum_{j=0}^{N} \lambda_j g_j \qquad (7)$$

where λ_j are the Lagrange multipliers. Then form

$$\frac{\partial L(a'_0, \lambda_j)}{\partial a'_0} = 0 \qquad (8)$$

and

$$\frac{\partial L(a'_0, \lambda_j)}{\partial \lambda_j} = 0, \ j = 0, 1, 2, \cdots, N \qquad (9)$$

Hereafter we refer to (8) as the Lagrange equation. We have in (8) and (9) a set of $1 + (N+1) = N + 2$ nonlinear equations involving the coefficients a'_i's and λ_j's as unknowns. Practically (8) is a single equation from which an expression for a'_0 can be derived in terms of the other a'_i's and λ_j's and then a'_0's can be substituted in (9). If the resulting nonlinear equations are solvable for a'_i's $(i = 0, 1, 2, \cdots, N)$ and hence for a'_0's and in this process if all λ_j's turn out to be positive, then the value $a'_0 = a_0^*$ will be the maximum. The corresponding polynomial $A'(z)$ with these a'_i's as coefficients will be stable or marginally stable. On the other hand if the nonlinear equations are not solvable, one can then conclude that this method fails to give us maximum value for a'_0 and hence we have failed to obtain a stable 1-D polynomial corresponding to the given unstable 1-D polynomial $A(z)$ by this approach.

Existence of maximum for 2-D QP and NSHP PLSI polynomials: Here, the stability of 2-D QP & NSHP PLSI polynomials is discussed.

Case I: QP PLSI polynomials: If the given 2-D QP polynomial $A(z_1, z_2)$ has a centrosymmetry in its coefficient matrix $[A]$, then it has been proved in general by Philippe Delsarte *et al*.[12] that the 2-D QP PLSI polynomial $B(z_1, z_2)$ will have its coefficient matrix $[B]$ symmetric.

In order to prove that the QP PLSI polynomial $B(z_1, z_2)$ is stable, we have to show or prove the existence of maximum for its constant term b_{00}. In this process, we arrive at a figure for the number of unknowns for the Mth degree 2-D QP polynomials. The total number of unknowns, namely b_{ij}'s is given by

$$(M+1)(M+1) = M^2 + 2M + 1 \qquad (10)$$

Since $[B]$ is symmetric, the number of independent b_{ij}'s is

$$\frac{(M+1)(M+2)}{2} = \frac{M^2 + 3M + 2}{2} \qquad (11)$$

So the number of λ_j's is

$$\frac{M^2 + 3M + 2}{2} \qquad (12)$$

Thus we have a total number of unknowns U as the sum of the two numbers given in (11) and (12). That is $U = M^2 + 3M + 2$. However, the total number of independent constraint equations will be

$$Q = (2M^2 + 2M + 1) - \frac{M(M+1)}{2} + 1$$

$$= \frac{3M^2 + 3M + 2}{2} + 1 \qquad (13)$$

The '+1' in equation (13) is due to the Lagrange equation. Theoretically the optimum for b'_{00} exists if $U \geq Q$. That is, if

$$M^2 + 3M + 2 \geq \frac{3M^2 + 3M + 2}{2} + 1$$

$$2M^2 + 6M + 4 \geq 3M^2 + 3M + 2 + 2$$

or

$$-M^2 + 3M \geq 0 \qquad (14)$$

So the maximum value of M for which (14) is satisfied is $M = 3$. Thus for any value of $M < 3$, there may exist a solution for the equations and hence the optimum for b'_{00} exists. This means that the PLSI polynomial $B(z_1, z_2)$ will be stable. And for any $M > 3$, the corresponding PLSI will be unstable.

But when $M = 3$, an interesting situation arises. That is, the number of equations is equal to the number of unknowns. If the number of unknowns is equal to the number of nonlinear equations, the real solution may not necessarily exist. In general if the number of equations is equal to the number of unknowns, the solution may or may not exist.

Case II: NSHP PLSI polynomials: If the given 2-D NSHP polynomial $A(z_1, z_2)$ has a centrosymmetry in its coefficient matrix [A], then it has been shown that the 2-D NSHP PLSI polynomial $B(z_1, z_2)$ will not have its coefficient matrix [B] symmetric[11].

In order to prove that the NSHP PLSI polynomial $B(z_1, z_2)$ is stable, we have to show or prove the existence of maximum for its constant term b_{00}. In this process, we first arrive at a figure for the number of unknowns for the Mth degree 2-D NSHP polynomials. For the 2-D NSHP polynomial of Mth degree, the total number of constraint equations are $4M^2 + 2M + 2$. But the number of unknowns λ_j's is $2M^2 + 2M + 1$ and b_{ij} is $2M^2 + 2M + 1$ and hence the total of

$4M^2 + 4M + 2$. (The highest order of the form preserving 1-D polynomial being $4M^2 + 2M$ for the Mth degree NSHP polynomial). Since $4M^2 + 4M + 2 > 4M^2 + 2M + 2$, the numbers of unknowns is more than the number of equations and it can easily be solved for and hence the optimum b_{00} exists. Therefore the PLSI polynomial is stable.

Numerical examples: Consider now the following 2-D QP 5th degree polynomial (in matrix form), which is unstable.

$$A(z_1, z_2) = [1 \ z_1 \ z_1^2 \ z_1^3 \ z_1^4 \ z_1^5]$$

$$\begin{bmatrix} 20.01 & 63.84 & 17.45 & 114.37 & -12.72 & -170.5 \\ 63.84 & 181.18 & -150.16 & -213.79 & -66.29 & 25.55 \\ 17.45 & -150.16 & -46.27 & 43.36 & -46.87 & 25.49 \\ 114.37 & -213.79 & 43.36 & 113.66 & -55.37 & -11.46 \\ -12.72 & -66.29 & -46.87 & -55.37 & 52.54 & 126.44 \\ -170.5 & 22.55 & 25.49 & -11.46 & 126.44 & -90.629 \end{bmatrix} \begin{bmatrix} 1 \\ z_2 \\ z_2^2 \\ z_2^3 \\ z_2^4 \\ z_2^5 \end{bmatrix}$$

As we notice, coefficient matrix of $A(z_1, z_2)$ is symmetric. We find that the coefficient matrix of the 2-D PLSI polynomial $B(z_1, z_2)$ is also symmetric. It is found that the form preserving 1-D polynomial of $B(z_1, z_2)$, i.e. $B(z^{11}, z)$ has a pair of complex conjugate zeros whose magnitude is 0.99388. [As the z-transform is defined with positive powers of z, for the PLSI polynomial to be stable, the magnitudes of all its zeros should be greater than 1]. So the PLSI in this case is unstable.

To verify this theoretically, we now use the Lagrange multiplier method as discussed earlier. As the coefficient matrix of the PLSI polynomial is also symmetric, the number of unknowns will be less than the number of equations in the process of optimization and hence the equations are not solvable in general. Therefore, we are unable to obtain the maximum for the constant term and the PLSI becomes unstable.

On the other hand, consider the following 2-D NSHP polynomial of degree 2

$$A(z_1, z_2) = 0.6 + 0.9z_2 + 0.3z_2^2 + 0.9z_1 + 1.5z_1z_2 + 0.9z_1z_2^2 +$$
$$0.3z_1^2 + 0.9z_1^2z_2 + 0.6z_1^2z_2^2 + 0.6z_1^2z_2^2 +$$
$$0.5z_1z_2^{-1} + 0.8z_1^2z_2^{-1} + 0.7z_1z_2^{-2} + z_1^2z_2^{-2} \qquad (15)$$

The PLSI $B(z_1, z_2)$ is found to be stable even though the original NSHP polynomial $A(z_1, z_2)$ has centrosymmetry among the coefficients in the quarter plane. To verify this theoretically, we have to use the Lagrange multiplier method. In this process, the total number of equations and unknowns turn out to be 22 and 26, respectively. As the number of unknowns is

more than the equations, the equations are solvable and hence the optimum exists. Thus the PLSI is found to be stable.

CONCLUSION

In the case of quarter-plane filters, if the given unstable polynomial $A(z_1, z_2)$ has a coefficient matrix [A] such that the resulting PLSI polynomial $B(z_1, z_2)$ has no relationship among themselves, the polynomial $B(z_1, z_2)$ is always stable when the degrees of $B(z_1, z_2)$ and $A(z_1, z_2)$ are one and the same. This can be easily guessed from the matrix [A] since no symmetry of any kind will ensure that there will not be any relationship among the coefficients of $B(z_1, z_2)$. On the other hand, if there is symmetry among the coefficients in the coefficient matrix [A] and hence in the PLSI [B], the resulting PLSI may not be stable if the degree is greater than two. This symmetry in quarter-plane polynomials therefore violates the Modified Shank's conjecture by Jury.

In nonsymmetric half-plane filters, even if the original polynomial $A(z_1, z_2)$ has a coefficient matrix [A] which is centrosymmetric or symmetric, the resulting PLSI polynomial $B(z_1, z_2)$ will not have any symmetry in its coefficient matrix and hence the PLSI will always be stable provided the degrees of both the polynomials are one and the same. Therefore the Modified Shanks' conjecture holds true for a special classes of 2-D polynomials i.e., NSHP polynomials.

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