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Measure for No Three-Factor Interaction Model in Three-Way Contingency Tables

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Abstract: Problem statement: For 2×2×K contingency tables, the measure is considered to represent the degree of departure from a log-linear model of No Three-Factor Interaction (NOTFI). We are interested in considering a similar measure for general I×J×K contingency tables. **Approach:** The present study proposed a measure to represent the degree of departure from the NOTFI model for I×J×K contingency tables. Also the approximate confidence interval for the proposed measure is given. **Results:** The proposed measure was applied and analyzed (1) for a 3×4×4 cross-classification data of dumping severity, hospital and operation which treat duodenal ulcer patients corresponding to removal of various amounts of the stomach and (2) for a 2×3×4 cross-classification data of experiment of animals (mouse and rat) on cancer (the tumor of leukemia and lymphoma) and tolazamide. **Conclusion:** The proposed measure is useful for comparing the degrees of departure from the NOTFI model in several tables.

Key words: Diversity index, odds-ratio, power-divergence

INTRODUCTION

For an I×J×K contingency table, let p_{ijk} denote the probability that an observation will fall in the (i, j, k) th cell of the table $(i = 1,\dots,I; j = 1,\dots,J; k = 1,\dots,K)$. One can express $\log p_{ijk}$ as:

$$\log p_{ijk} = u + u_{1(i)} + u_{2(j)} + u_{3(k)} + u_{12(ij)} + u_{13(ik)} + u_{23(jk)} + u_{123(ijk)}$$

Where:

$$\sum_i u_{s(i)} = 0 \quad (s = 1,2,3)$$

$$\sum_i u_{st(ij)} = \sum_j u_{st(ij)} = 0 \quad (1 \le s < t \le 3)$$

$$\sum_i u_{123(ijk)} = \sum_j u_{123(ijk)} = \sum_k u_{123(ijk)} = 0$$

e.g., Bishop *et al*. (1975). Then the No Three-Factor Interaction (NOTFI) model is defined by setting the parameters as:

$$u_{123(ijk)} = 0$$

for all i, j, k. This model can also be expressed as:

$$\theta_{ij(1)} = \cdots = \theta_{ij(K)} \quad (i = 1,\dots,I-1; j = 1,\dots,J-1)$$

Where:

$$\theta_{ij(t)} = \frac{p_{ijt}p_{i+1,j+1,t}}{p_{i,j+1,t}p_{i+1,j,t}}$$

e.g., Agresti (1984). When the NOTFI model does not hold, we are interested in measuring the degree of departure from the NOTFI model, i.e., the degree of non-uniformity of odds-ratios $\{\theta_{ij(t)}\}$.

For the 2×2×K contingency table, namely, when I = J = 2, Tomizawa (1993) and Yamamoto *et al*. (2008) considered measures which represent the degree of departure from the NOTFI model.

The purpose of present research is to extend these measures into the I×J×K table. The extended measure

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would be useful for comparing the degrees of departure from the NOTFI model in several tables.

MATERIALS AND METHODS

An extended measure: Consider the I×J×K contingency table. Let:

$$D_{ij} = \sum_{k=1}^{K} \theta_{ij(k)}, \quad \theta^*_{ij(t)} = \frac{\theta_{ij(t)}}{D_{ij}}$$
$$(i = 1,\dots,I-1; j = 1,\dots,J-1; t = 1,\dots,K)$$

Assuming that the $\{p_{ijk}\}$ are positive, consider a measure to represent the degree of departure from the NOTFI model, defined by:

$$\Psi^{(\lambda)} = \frac{1}{\delta^*} \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \delta_{ij} \varphi_{ij}^{(\lambda)} \quad (\lambda > -1)$$

Where:

$$\delta_{ij} = \sum_{l=i}^{i+1} \sum_{m=j}^{j+1} \sum_{n=1}^{K} p_{lmn}$$

$$\delta^* = \sum_{s=1}^{I-1} \sum_{t=1}^{J-1} \delta_{st}$$

$$\varphi_{ij}^{(\lambda)} = 1 - \frac{H_{ij}^{(\lambda)}(\theta^*)}{C^{(\lambda)}}$$

$$H_{ij}^{(\lambda)}(\theta^*) = \frac{1}{\lambda}\left(1 - \sum_{t=1}^{K} (\theta^*_{ij(t)})^{\lambda+1}\right)$$

$$C^{(\lambda)} = \frac{1}{\lambda}\left[1 - \left(\frac{1}{K}\right)^{\lambda}\right]$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$. Note that λ is a real value that is chosen by the user. The submeasure $\varphi_{ij}^{(\lambda)}$ represents the degree of non-uniformity of odds-ratios $\{\theta_{ij(t)}\}$ for fixed i and j. Note that $H_{ij}^{(\lambda)}(\theta^*)$ is Patil and Taillie (1982) diversity index of degree λ for $\{\theta^*_{ij(t)}\}$, t = 1,…,K, which includes the Shannon entropy (when $\lambda = 0$) in a special case. When I = J = 2, the measure $\Psi^{(\lambda)}$ is identical with the measure in Yamamoto *et al*. (2008) and when I = J = 2 and $\lambda = 0$, it is identical with the measure in Tomizawa (1993). The submeasure $\varphi_{ij}^{(\lambda)}$ may be expressed as:

$$\varphi_{ij}^{(\lambda)} = \frac{\lambda+1}{K^{\lambda} C^{(\lambda)}} I_{ij}^{(\lambda)}\left(\left\{\theta^*_{ij(t)}\right\}; \left\{\frac{1}{K}\right\}\right)$$

Where:

$$I_{ij}^{(\lambda)}(\cdot;\cdot) = \frac{1}{\lambda(\lambda+1)} \sum_{t=1}^{K} \theta^*_{ij(t)} \left[\left(\frac{\theta^*_{ij(t)}}{1/K}\right)^{\lambda} - 1\right]$$

Note that $I_{ij}^{(\lambda)}(\{\theta^*_{ij(t)}\};\{\frac{1}{K}\})$ is the power-divergence between $\{\theta^*_{ij(t)}\}$ and $\{\frac{1}{K}\}$, which includes the Kullback-Leibler information (when $\lambda = 0$) in a special case. For more details of the power-divergence, Cressie and Read (1984) and Read and Cressie (1988).

The $H_{ij}^{(\lambda)}(\theta^*)$ must lie between 0 and $C^{(\lambda)}$ but it cannot attain the lower limit of 0 in terms of the assumption that are $\{p_{ijk}\}$ positive. Thus the submeasure $\varphi_{ij}^{(\lambda)}$ must lie between 0 and 1 and therefore the measure $\Psi^{(\lambda)}$ must lie between 0 and 1, but it cannot attain the upper limit of 1. Now it is easily seen that for each $\lambda(>-1)$, the NOTFI model holds if and only if the $\varphi_{ij}^{(\lambda)} = 0$ for every i = 1,…,I-1; j = 1,…,J-1, i.e., $\Psi^{(\lambda)} = 0$. According to the weighted sum of the diversity index or the power-divergence, $\Psi^{(\lambda)}$ represents the degree of departure from NOTFI model and the degree increases as the value of $\Psi^{(\lambda)}$ increases.

Approximate confidence interval for measure: Let n_{ijk} denote the observed frequency in the (i, j, k)th cell of the I×J×K table (i = 1,…,I; j = 1,…,J; k = 1,…,K). Assuming that $\{n_{ijk}\}$ result from full multinomial sampling, we shall consider an approximate standard error and large-sample confidence interval of the measure $\Psi^{(\lambda)}$, using the delta method of which descriptions are given by, for example, Bishop *et al*. (1975). The sample version of measure $\Psi^{(\lambda)}$, i.e., $\hat{\Psi}^{(\lambda)}$, is given by $\Psi^{(\lambda)}$ with $\{p_{ijk}\}$ replaced by $\{\hat{p}_{ijk}\}$, where $\hat{p}_{ijk} = n_{ijk}/n$ and $n = \sum\sum\sum n_{ijk}$. Using the delta method, $\sqrt{n}\left(\hat{\Psi}^{(\lambda)} - \Psi^{(\lambda)}\right)$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance:

$$\sigma^2 = \frac{1}{(\delta^*)^2} \times \left[\sum_{k=1}^{I} \sum_{l=1}^{J} \sum_{m=1}^{K} (w_{klm}^{(\lambda)})^2 p_{klm} - \left(\sum_{k=1}^{I} \sum_{l=1}^{J} \sum_{m=1}^{K} w_{klm}^{(\lambda)} p_{klm}\right)^2\right]$$

with:

$$w_{klm}^{(\lambda)} = \varphi_{k-1,l-1}^{(\lambda)} + \varphi_{k-1,l}^{(\lambda)} + \varphi_{k,l-1}^{(\lambda)} + \varphi_{kl}^{(\lambda)} - \Delta_{kl}\Psi^{(\lambda)}$$
$$+ \frac{1}{p_{klm}}\left(A_{k-1,l-1(m)}^{(\lambda)} - A_{k-1,l(m)}^{(\lambda)} - A_{k,l-1(m)}^{(\lambda)} + A_{kl(m)}^{(\lambda)}\right)$$

$$A^{(\lambda)}_{st(m)} = \frac{(\lambda+1)\delta_{st}\theta_{st(m)}}{\lambda C^{(\lambda)} D_{st}^{\lambda+2}} \times \left[D_{st}(\theta_{st(m)})^{\lambda} - \sum_{u=1}^{K} (\theta_{st(u)})^{\lambda+1} \right]$$

$$\varphi^{(\lambda)}_{s0} = \varphi^{(\lambda)}_{sJ} = \varphi^{(\lambda)}_{0t} = \varphi^{(\lambda)}_{It} = 0$$

$$A^{(\lambda)}_{s0(m)} = A^{(\lambda)}_{sJ(m)} = A^{(\lambda)}_{0t(m)} = A^{(\lambda)}_{It(m)} = 0$$

$$(s = 0,1,\ldots,I; t = 0,1,\ldots,J)$$

$$\Delta_{st} = \begin{cases} 1 & (s=1;\ t=1), \\ 1 & (s=1;\ t=J), \\ 1 & (s=I;\ t=1), \\ 1 & (s=I;\ t=J), \\ 2 & (s=1,I;\ t=2,\ldots,J-1), \\ 2 & (s=2,\ldots,I-1;\ t=1,J), \\ 4 & (s=2,\ldots,I-1;\ t=2,\ldots,J-1) \end{cases}$$

Let $\hat{\sigma}^2$ denote σ^2 with $\{p_{ijk}\}$ replaced by $\{\hat{p}_{ijk}\}$. Then $\hat{\sigma}/\sqrt{n}$ is an estimated approximate standard error for $\hat{\Psi}^{(\lambda)}$ and $\hat{\Psi}^{(\lambda)} \pm z_{p/2}\hat{\sigma}/\sqrt{n}$ is an approximate 100(1-p) percent confidence interval for $\Psi^{(\lambda)}$, where $z_{p/2}$ is the percentage point from the standard normal distribution corresponding to a two-tail probability equal to p.

RESULTS

Example 1: The data in Table 1a, taken from Grizzle *et al*. (1969), are a 3×4×4 cross-classification of dumping severity, hospital and operation (Agresti, 1984; Tomizawa, 1992). Also, Table 1b rearranges the data in Table 1a. Four different operations for treating duodenal ulcer patients correspond to removal of various amounts of the stomach. Operation A is drainage and vagotomy, B is 25% resection (antrectomy) and vagotomy, C is 50% resection (hemigastrectomy) and vagotomy and D is 75% resection. The dumping severity variable describes the extent of an undesirable potential consequence of the operation.

The NOTFI model indicates (1) the odds ratios (association) between the dumping severity and hospital are uniform among the operations and (2) the odds ratios (association) between the dumping severity and operation are uniform among the hospitals. For these data, we are now interested in two kinds of the degrees of departure from the NOTFI model; namely, (1) what degree the odds ratios (association) between the dumping severity and hospital are apart from the uniformity among the operations and (2) what degree the odds ratios (association) between the dumping severity and operation are apart from the uniformity among the hospitals.

We see from Table 2 that the estimated value of measure $\Psi^{(\lambda)}$ for (1) is different (though it is slight) from that for (2). In addition, we see that the degree of departure from the uniformity of odds ratios between the dumping severity and hospital among the operations is somewhat greater than the degree of departure from the uniformity of odds ratios between the dumping severity and operation among the hospitals.

Example 2: The data in Table 3, taken from Yanagawa (1986), are 2×3×4 cross-classification of experiment on animal for cancer according to the tolazamide (control, lower dose and higher dose), the tumor of leukemia and lymphoma and the animals (female mouse, male mouse, female rat and male rat).

Table 1: Cross-classification of duodenal ulcer patients according to dumping severity, hospital and operation; taken from Grizzle *et al*. (1969)

		Hospital			
Operation	Dumping severity	1	2	3	4
(a) Observations					
A	N	23	18	8	12
	S	7	6	6	9
	M	2	1	3	1
B	N	23	18	12	15
	S	10	6	4	3
	M	5	2	4	2
C	N	20	13	11	14
	S	13	13	6	8
	M	5	2	2	3
D	N	24	9	7	13
	S	10	15	7	6
	M	6	2	4	4
		Operation			
Hospital		A	B	C	D
(b) Table rearranged Table 1a					
1	N	23	23	20	24
	S	7	10	13	10
	M	2	5	5	6
2	N	18	18	13	9
	S	6	6	13	15
	M	1	2	2	2
3	N	8	12	11	7
	S	6	4	6	7
	M	3	4	2	4
4	N	12	15	14	13
	S	9	3	8	6
	M	1	2	3	4

Note: N: None; S: Slight; M: Moderate

Table 2: Estimates of $\Psi^{(\lambda)}$, estimated approximate standard error for $\hat{\Psi}^{(\lambda)}$, approximate 95% confidence interval for $\Psi^{(\lambda)}$, applied to Table 1a and 1b

Values of λ	Estimated measure	Standard error	Confidence interval
(a) For Table 1a			
-0.4	0.074	0.051	(-0.026, 0.174)
0	0.095	0.066	(-0.034, 0.223)
0.6	0.100	0.072	(-0.041, 0.241)
1.0	0.093	0.070	(-0.044, 0.231)
1.6	0.077	0.063	(-0.046, 0.200)
(b) For Table 1b			
-0.4	0.054	0.042	(-0.029, 0.136)
0	0.067	0.053	(-0.037, 0.171)
0.6	0.068	0.056	(-0.041, 0.178)
1.0	0.062	0.053	(-0.041, 0.165)
1.6	0.048	0.045	(-0.039, 0.136)

Table 3: Cross-classification of experiment on animal for cancer according to tolazamide, tumor and animal; taken from Yanagawa (1986)

		Tolazamide		
Animal	Tumor	Control	Lower dose	Higher dose
Female	No	9	31	30
Mouse	Yes	6	2	4
Male	No	10	30	33
Mouse	Yes	4	5	1
Female	No	11	30	33
Rat	Yes	4	3	2
Male	No	13	34	31
Rat	Yes	2	1	4

Table 4: Estimates of $\Psi^{(\lambda)}$, estimated approximate standard error for $\hat{\Psi}^{(\lambda)}$, approximate 95% confidence interval for $\Psi^{(\lambda)}$, applied to Table 3

Values of λ	Estimated measure	Standard error	Confidence interval
-0.4	0.182	0.141	(-0.095, 0.459)
0	0.215	0.175	(-0.128, 0.558)
0.6	0.211	0.199	(-0.179, 0.601)
1.0	0.192	0.205	(-0.209, 0.594)
1.6	0.158	0.202	(-0.239, 0.554)

The NOTFI model indicates that the odds ratios (association) between the dose of tolazamide and the tumor are uniform among the animals. For these data, we are now interested in the degree of departure from the NOTFI model; namely what degree the odds ratios (association) between the dose of tolazamide and the tumor are apart from the uniformity among the animals.

Table 4 shows the degree of departure from the uniformity of odds ratios between the dose of tolazamide and the tumor among the four kinds of animals. We see from Table 2 and 4 that the degree of departure from the NOTFI model is greater for the data in Table 3 than for the data in Table 1.

Table 5: Values of power-divergence statistic $W^{(\lambda)}$ for testing goodness-of-fit of the NOTFI model applied to Table 1a, 1b and 3

Values of λ	For Table 1a	For Table 1b
(a) For Table 1a and 1b with 18 degrees of freedom		
-0.4	12.50	12.50
0	12.50	12.50
0.6	12.56	12.56
1.0	12.64	12.64
1.6	12.82	12.82
Values of λ		For Table 3
(b) For Table 3 with 6 degrees of freedom		
-0.4		7.473
0		7.322
0.6		7.264
1.0		7.331
1.6		7.589

DISCUSSION

The readers may be interested in the relation between the measure and the test statistic for goodness-of-fit of the NOTFI model. Let $W^{(\lambda)}$ denote the power-divergence statistic for testing goodness-of-fit of the NOTFI model with (I-1)(J-1)(K-1) degrees of freedom, i.e.:

$$W^{(\lambda)} = \frac{2}{\lambda(\lambda+1)} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} n_{ijk} \left[\left(\frac{n_{ijk}}{\hat{m}_{ijk}} \right)^{\lambda} - 1 \right] \quad (-\infty < \lambda < \infty)$$

where $\hat{m}_{ijk}$ is the maximum likelihood estimate of the expected frequency m_{ijk} under the NOTFI model and the values at $\lambda = -1$ and $\lambda = 0$ are taken to be the limits as $\lambda \to -1$ and $\lambda \to 0$, respectively

For the details of power-divergence test statistic, Cressie and Read (1984) and Read and Cressie (1988). In particular, note that $W^{(0)}$ and $W^{(1)}$ are the likelihood ratio and Pearson chi-squared statistics, respectively. Table 5 gives the values of $W^{(\lambda)}$ applied to the data in Tables 1a, 1b and 3. We point out that the value of $W^{(\lambda)}$ for Table 1a is theoretically equal to that for Table 1b though the value of $\hat{\Psi}^{(\lambda)}$ for Table 1a is not equal to that for Table 1b.

Therefore it would not be appropriate to use the test statistic $W^{(\lambda)}$ for measuring and comparing the degree of non-uniformity of odds ratios in several tables and the users should use the measure $\hat{\Psi}^{(\lambda)}$.

CONCLUSION

For the I×J×K contingency table, denote the three variables by X, Y and Z. The NOTFI model indicates

Table 6: (a), (b) Artificial data (n is sample size) and (c) corresponding values of odds-ratios $\{\theta_{ij(t)}\}$ for Tables 6a and 6b

		Y		
Z	X	(1)	(2)	(3)
(a) n = 207				
(1)	(1)	5	4	5
	(2)	5	8	4
	(3)	4	8	5
(2)	(1)	9	3	6
	(2)	6	6	3
	(3)	3	6	9
(3)	(1)	6	4	3
	(2)	4	3	6
	(3)	3	6	4
(4)	(1)	10	8	5
	(2)	8	5	10
	(3)	5	10	8
(b) n = 1035				
(1)	(1)	25	20	25
	(2)	25	40	20
	(3)	20	40	25
(2)	(1)	45	15	30
	(2)	30	30	15
	(3)	15	30	45
(3)	(1)	30	20	15
	(2)	20	15	30
	(3)	15	30	20
(4)	(1)	50	40	25
	(2)	40	25	50
	(3)	25	50	40

		j	
t	i	1	2
(c) Values of $\{\theta_{ij(t)}\}$ for Tables 6a and 6b			
1	1	2.00	0.40
	2	1.25	1.25
2	1	3.00	0.25
	2	2.00	3.00
3	1	1.13	2.67
	2	2.67	0.33
4	1	0.78	3.20
	2	3.20	0.40

that (1) each of (I-1)(J-1) odds ratios between X and Y is uniform among Z, (2) each of (I-1)(K-1) odds ratios between X and Z is uniform among Y and (3) each of (J-1)(K-1) odds ratios between Y and Z is uniform among X. The measure $\Psi^{(\lambda)}$ proposed in this study is useful for measuring and comparing the three kinds of degrees of departure from the NOTFI model; namely, (1) what degree the odds ratios between X and Y are apart from the uniformity among Z, (2) what degree the odds ratios between X and Z are apart from the uniformity among Y and (3) what degree the odds ratios between Y and Z are apart from the uniformity among X.

Table 7: Values of $\hat{\Psi}^{(\lambda)}$ applied to Table 6a and 6b

Values of λ	For Table 6a	For Table 6b
-0.4	0.134	0.134
0	0.163	0.163
0.6	0.162	0.162
1.0	0.147	0.147
1.6	0.118	0.118

Table 8: Values of power-divergence statistic $W^{(\lambda)}$ (with 12 degrees of freedom) for testing goodness-of-fit of the NOTFI model, applied to Tables 6a and 6b

Values of λ	For Table 6a	For Table 6b
-0.4	8.586	42.930
0	8.499	42.495
0.6	8.421	42.105
1.0	8.401	42.005
1.6	8.417	42.085

From Example 1, we have seen using the proposed measure $\hat{\Psi}^{(\lambda)}$ that for the data in Table 1, the degree of departure from the uniformity of odds ratios (association) between the dumping severity and hospital among the operations is somewhat greater than the degree of departure from the uniformity of odds ratios (association) between the dumping severity and operation among the hospitals. In addition, from Examples 1 and 2, we have seen that the degree of departure from the uniformity of odds ratios (association) between the dose of tolazamide and the tumor among the animals for the data in Table 3 is greater than the degree of departure from the uniformity of odds ratios (association) for the data in Table 1.

The measure $\hat{\Psi}^{(\lambda)}$ would be useful for comparing the degrees of departure from the NOTFI model in several tables. Consider the artificial data in Table 6a and 6b. All values of observed frequencies in Table 6a multiplied by 5 equal the values in Table 6b. Thus, it is natural that the estimated odds-ratios between variables X and Y at each level of Z for Table 6b are equal to those for Table 6a (Table 6c). Therefore, the value of $\hat{\Psi}^{(\lambda)}$ (for every λ) for Table 6a is identical with that for Table 6b (Table 7). However the value of $W^{(\lambda)}$ is greater for Table 6b than for Table 6a (Table 8). Therefore the measure $\hat{\Psi}^{(\lambda)}$ rather than test statistic $W^{(\lambda)}$ would be useful for comparing the degrees of departure from the NOTFI model in several tables.

The readers may be interested in which value of λ is preferred for a given table. However, in comparing tables, it seems difficult to discuss this. For example, consider the artificial data in Table 9a and 9b. We see from Table 9c that the value of $\hat{\Psi}^{(0)}$ is greater for Table

9a than for Table 9b, but the value of $\hat{\Psi}^{(1)}$ is less for Table 9a than for Table 9b. So, for these cases, it may be impossible to decide (by using $\hat{\Psi}^{(\lambda)}$) whether the degree of departure from the NOTFI model is greater for Table 9a or for Table 9b. But generally, for the comparison between two tables, it would be possible to draw a conclusion if $\hat{\Psi}^{(\lambda)}$ (for every λ) is always greater (or always less) for one table than for the other table. Thus, it seems to be important that the analyst calculates the value of $\hat{\Psi}^{(\lambda)}$ for various values of λ and discusses the degree of departure from the NOTFI model in terms of $\hat{\Psi}^{(\lambda)}$ values.

The measure $\hat{\Psi}^{(\lambda)}$ would be useful when one wants to measure how far the odds-ratios $\{\theta_{ij(t)}\}$ are directly distant from the uniformity, although $W^{(\lambda)}/n$ may be useful when one wants to see how far the estimated cell probability distribution with the structure of NOTFI is distant from the sample cell probability distribution.

Table 9: (a), (b) Artificial data (n is sample size) and (c) corresponding values of $\hat{\Psi}^{(\lambda)}$ applied to Tables 9a and 9b

		Y		
Z	X	(1)	(2)	(3)
(a) n = 1455				
(1)	(1)	15	35	35
	(2)	20	75	65
	(3)	20	40	60
(2)	(1)	20	35	20
	(2)	20	55	70
	(3)	25	55	65
(3)	(1)	20	35	25
	(2)	30	75	50
	(3)	15	55	60
(4)	(1)	10	15	20
	(2)	20	110	70
	(3)	10	70	35
(b) n = 1425				
(1)	(1)	15	25	30
	(2)	25	65	75
	(3)	10	55	55
(2)	(1)	15	35	25
	(2)	10	75	70
	(3)	15	45	75
(3)	(1)	10	35	20
	(2)	15	70	75
	(3)	15	55	65
(4)	(1)	5	20	25
	(2)	20	65	70
	(3)	25	50	65

Values of λ	For Table 9a	For Table 9b
(c) Values of $\hat{\Psi}^{(\lambda)}$		
-0.4	0.051	0.050
0	0.066	0.065
0.6	0.069*	0.070
1.0	0.064*	0.066
1.6	0.051*	0.055

*: Indicates that $\hat{\Psi}^{(\lambda)}$ is less for Table 9a than for Table 9b

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