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A Method of Deductive Logical Inference Proofs

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Abstract: A method of proof is presented and used in proving theorems in logic and set theory. Many commonly used methods of proof are rigid and not easy to apply in proving different theorems. This study deductively draws conclusions from rules of logical inference and then, it generalizes the deduction methods to be applied to logic and set theory. Then, it shows how this method of logical inference can be used to prove implications involving conjunction or disjunction of premises and to prove some identities in set theory involving implication or containment.

Key words: Deductive inference, conjunction, disjunction, implication, containment, method

INTRODUCTION

Logical inference aims at forming a conclusion based on some given premises. Generally, deductive inference involves conditional reasoning that follows the (if *A* then *B*) format to obtain valid conclusions from true premises.

Deductive reasoning is sometimes used in automated inference systems. A recent study[1] employed deductive reasoning to present a model for producing proof presentations from machine oriented inference structures. A book was written to present fundamental theoretical results concerning inference rules in deductive formal systems [3].

In addition to the well-known applications in set theory, computer logic design and artificial intelligence, deductive logical inference has many applications in psychology[2,4,5].

This study shows how to simplify rules of logical inference and then how to apply these ideas to obtain deductive logical inference proofs in logic and in set theory.

SIMPLIFYING LOGICAL INFERENCE RULES

This study shows a different method for simplifying logical inference rules and proving theorems.

A simple premise may be used to obtain several valid conclusions. This is based on the following idea.

Assume P is a premise and Q is a conclusion. Table 1 shows the possible values of $P \rightarrow Q$ for different values of P and Q. Based on this table, it can be seen that if the conclusion, Q, is true, then the implication $P \rightarrow Q$ is true. In addition, if the premise P is false, then the implication $P \rightarrow Q$ is true.

Table 1: The possible values of implication

P	Q	$P \rightarrow Q$
False	False	True
False	True	True
True	False	False
True	True	True

Let x and y be arbitrary Boolean variables. Each of the following simple premises is used with logical inference to draw multiple conclusions.

An asserted variable as a premise: Assuming x is true then the disjunction $(x \vee y)$ is true and the implication $(y \rightarrow x)$ is true. Therefore, these implications are valid:

$$x \rightarrow (x \vee y)$$
$$x \rightarrow (y \rightarrow x)$$

A negated variable as a premise: Assuming $\sim x$ (negation of x) is true then $(x \rightarrow y)$ is true and $\sim(x \wedge y)$ is true. Therefore, the following implications are valid:

$$\sim x \rightarrow (x \rightarrow y)$$
$$\sim x \rightarrow (x \uparrow y) \text{ , where } (x \uparrow y) \text{ denotes } \sim(x \wedge y).$$

Conjunction as a premise: When $(x \wedge y)$ is true then x is true and y is true, Table 2. Therefore, $(x \vee y)$ is true and the implications $(x \rightarrow y)$ and $(y \rightarrow x)$ are both true. Consequently, the following implications are valid:

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$(x \wedge y) \rightarrow x$
$(x \wedge y) \rightarrow y$
$(x \wedge y) \rightarrow (x \vee y)$
$(x \wedge y) \rightarrow (y \rightarrow x)$
$(x \wedge y) \rightarrow (x \rightarrow y)$

Table 2: The possible values of conjunction

x	y	$x \wedge y$
False	False	False
False	True	False
True	False	False
True	True	True

Table 3: The possible values of NOR

x	y	$x \vee y$	$x \downarrow y$
False	False	False	True
False	True	True	False
True	False	True	False
True	True	True	False

Table 4: The possible values of Exclusive-OR and Exclusive-NOR

x	y	$x \oplus y$	$x \leftrightarrow y$
False	False	False	True
False	True	True	False
True	False	True	False
True	True	False	True

The last two implications result in the valid implication:

$(x \wedge y) \rightarrow (x \leftrightarrow y)$.

NOR as a premise: Suppose $(x \downarrow y)$ is true where $\downarrow$ denotes NOR. Then, both x and y must be false, Table 3. This means that $\sim x$ and $\sim y$ must both be true and so must be $(x \uparrow y)$.

Furthermore, $(x \rightarrow y)$ and $(y \rightarrow x)$ are true, implying that $(x \leftrightarrow y)$ is also true. Consequently, the following implications are valid:

$(x \downarrow y) \rightarrow \sim x$
$(x \downarrow y) \rightarrow \sim y$
$(x \downarrow y) \rightarrow (x \uparrow y)$
$(x \downarrow y) \rightarrow (x \rightarrow y)$
$(x \downarrow y) \rightarrow (y \rightarrow x)$
$(x \downarrow y) \rightarrow (x \leftrightarrow y)$.

Exclusive-OR as a premise: Suppose $(x \oplus y)$ is true where $\oplus$ denotes Exclusive-OR and the possible values of Exclusive-OR are shown in Table 4. Then, $(x \vee y)$ is true and $(x \uparrow y)$ is true. Therefore, the following implications are valid:

$(x \oplus y) \rightarrow (x \vee y)$
$(x \oplus y) \rightarrow (x \uparrow y)$.

Exclusive-NOR as a premise: Suppose $(x \leftrightarrow y)$ is true where $\leftrightarrow$ denotes Exclusive-NOR which is the same as logical equivalence. The possible values of Exclusive-NOR are shown in Table 4.

Then, $(x \rightarrow y)$ and $(y \rightarrow x)$ are both true. Therefore, the following implications are valid:

$(x \leftrightarrow y) \rightarrow (x \rightarrow y)$
$(x \leftrightarrow y) \rightarrow (y \rightarrow x)$.

DEDUCTIVE LOGICAL INFERENCE PROOFS

The methods used above can be generalized to prove some theorems, especially those involving conjunction or disjunction of premises.

Here, two well-known theorems are proven in a way different from the classical method found in textbooks.

Theorem 1: To show that $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q$ is true, it suffices to show that $(P_i \rightarrow Q)$ is true for any i, $1 \leq i \leq n$.

Normally, this theorem may be proven as follows:

$(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q$
$\Leftrightarrow \sim (P_1 \wedge P_2 \wedge \ldots \wedge P_n) \vee Q$
$\Leftrightarrow (\sim P_1 \vee \sim P_2 \vee \ldots \vee \sim P_n) \vee Q$
$\Leftrightarrow (\sim P_1 \vee Q) \vee (\sim P_2 \vee Q) \vee \ldots \vee (\sim P_n \vee Q)$
$\Leftrightarrow (P_1 \rightarrow Q) \vee (P_2 \rightarrow Q) \vee \ldots \vee (P_n \rightarrow Q)$.

However, a different proof of this theorem may be obtained as follows. Take an arbitrary $(P_i \rightarrow Q)$ as a premise, where $1 \leq i \leq n$ and show that if it is true, then the implication $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q$ must be true as well and therefore, the argument:

$$\frac{P_i \rightarrow Q}{(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q} \text{ is valid.}$$

Proof: When $(P_i \rightarrow Q)$ is true, there are three cases for the values of P_i and Q, shown in Table 5. In both of Case 1 and Case 2, the value of Q is true and, therefore, the implication $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q$ is true. In Case 3, P_i is false, making the conjunction $(P_1 \wedge P_2 \wedge \ldots \wedge P_n)$ false. Since Q is also false in this case, the implication $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q$ is (False $\rightarrow$ False), which is true. Consequently, the argument is valid and the theorem is proven.

Similarly, it can be shown how an alternate proof can be obtained for Theorem 2, given below.

Theorem 2: To show that $(P_1 \vee P_2 \vee \ldots \vee P_n) \rightarrow Q$ is true, it is necessary to show that $(P_i \rightarrow Q)$ is true for all i, $1 \leq i \leq n$.

Table 5: Cases of P_i and Q when ($P_i \to Q$) is true

Case	P_i	Q	$P_i \to Q$
1	True	True	True
2	False	True	True
3	False	False	True

A common proof (Proof by Cases) is as follows:

$$(P_1 \vee P_2 \vee \ldots \vee P_n) \to Q$$
$$\Leftrightarrow \sim(P_1 \vee P_2 \vee \ldots \vee P_n) \vee Q$$
$$\Leftrightarrow (\sim P_1 \wedge \sim P_2 \wedge \ldots \wedge \sim P_n) \vee Q$$
$$\Leftrightarrow (\sim P_1 \vee Q) \wedge (\sim P_2 \vee Q) \wedge \ldots \wedge (\sim P_n \vee Q)$$
$$\Leftrightarrow (\sim P_1 \to Q) \wedge (\sim P_2 \to Q) \wedge \ldots \wedge (P_n \to Q).$$

However, a different proof of this theorem can be obtained as follows. Take an arbitrary ($P_i \to Q$) as a premise, where $1 \le i \le n$ and show that if it is true, then the implication $(P_1 \vee P_2 \vee \ldots \vee P_n) \to Q$ can be true or false and therefore, the argument:

$$\frac{P_i \to Q}{(P_1 \vee P_2 \vee \ldots \vee P_n) \to Q} \text{ is not valid.}$$

Proof: When ($P_i \to Q$) is true, there are three cases for the values of P_i and Q, shown in Table 5. In both of Case 1 and Case 2, the value of Q is true and, therefore, the implication $(P_1 \vee P_2 \vee \ldots \vee P_n) \to Q$ is true. In Case 3, where P_i is false, it follows that the disjunction $(P_1 \vee P_2 \vee \ldots \vee P_n)$ may be true or false and therefore, the implication $(P_1 \vee P_2 \vee \ldots \vee P_n) \to Q$ can be true or false. This makes the argument invalid and it is necessary to show that ($P_i \to Q$) is true for all values of i and the theorem is proven.

USING DEDUCTIVE LOGICAL INFERENCE RULES IN SET THEORY

The ideas used above may be applied to set theory. Many set theory laws can be proven deductively, especially laws involving implications. Containment and equality of sets, for example, can be expressed with implication as follows, where A and B are sets:

$$A \subset B \Leftrightarrow \forall x [x \in A \to x \in B] \text{ and}$$
$$A = B \Leftrightarrow \forall x [x \in A \leftrightarrow x \in B]$$
$$\Leftrightarrow \forall x [x \in A \to x \in B] \wedge \forall x [x \in B \to x \in A]$$
$$\Leftrightarrow [A \subset B \wedge B \subset A].$$

As an example, it is shown how to prove two such identities.

Consider the simple identity: $A - B \subset A$: This identity may be expressed with the following implication:

Table 6: Some premises and their conclusions

Premise	Conclusions
x	$(x \vee y)$, $(y \to x)$
~x	$(x \to y)$, $(x \uparrow y)$
$(x \wedge y)$	x, y, $(x \vee y)$, $(y \to x)$, $(x \to y)$, $(x \leftrightarrow y)$
$(x \downarrow y)$	~x, ~y, $(x \uparrow y)$, $(x \to y)$, $(y \to x)$, $(x \leftrightarrow y)$
$(x \oplus y)$	$(x \vee y)$, $(x \uparrow y)$
$(x \leftrightarrow y)$	$(x \to y)$, $(y \to x)$

$$(x \in A \wedge x \notin B) \Rightarrow x \in A,$$

and with an argument as follows:

$$\frac{\begin{array}{c} x \in A \\ x \notin B \end{array}}{x \in A}.$$

As shown with conjunction as a premise, $x \in A$ in the conclusion must always be true, and therefore, the argument is valid.

Consider the identity: If $A \subset B$, then $A \cap B = A$.

As an implication, this identity may be expressed as:

$$(x \in A \to x \in B) \Rightarrow [(x \in A \wedge x \in B) \leftrightarrow x \in A]$$

and with an argument as:

$$\frac{x \in A \to x \in B}{(x \in A \wedge x \in B) \leftrightarrow x \in A}.$$

Proof: The premise $(x \in A \to x \in B)$ has three cases, as shown in Table 5, with $(x \in A)$ as P_i and $(x \in B)$ as Q. In all of these cases, the conclusion $(x \in A \wedge x \in B) \leftrightarrow x \in A$ is always true and therefore, the argument is valid.

CONCLUSIONS

Deductive logical inference was used to obtain rules and then the methods of obtaining these rules were generalized to obtain different proofs of some theorems in logic and set theory.

- The main logical inference rules are summarized in Table 6, where each premise was used to obtain the listed conclusions.
- A new technique has been employed to prove implications that have premises consisting of conjunctions or disjunctions.
- It has been shown how logical inference can be used to prove some identities in set theory that involve implication or containment.

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