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A Theorem of Hunt for Semidynamical Systems

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Abstract: The main result of this paper provides a direct proof of a Hunt's Theorem of the classical potential theory for the so called local semidynamical systems in non locally compact infinite dimensional spaces.

Key words: Excessive Functions, Hitting Time, Reduite Operator, Semidynamical System

INTRODUCTION

The structure of this paper is adapted to the classical results which present a difficulty for the measurability. Starting by a transient semidynamical system $(X, \mathcal{B}, \Phi, \omega)$, [1-4], we introduce the notion of hitting time T_A of a measurable subset A of X_0. We prove the measurability of T_A with respect to the σ-algebra $\mathcal{B}_0(\Lambda)$ defined in [5] and we give further properties by using the semidynamical specificity. We give a relationship between the hitting time T_A and the reduite operator R^A in classical potential theory and we give a direct proof of a Hunt's Theorem [6]. Here we don't use the Choquet capacity [7, 8]. Howover in the work of Hunt [6] cited here by as principal reference, the measurability is ensured with respect to the initial σ-algebra, but in this work, we express the measurability in a weak sense i.e. the measurability along the trajectories. Which is sufficient for integrate with respect to the reference Lebesgue measure Λ.

Preliminary: Here, we will introduce some definitions which will be useful in the remainder of this study [1, 4, 5, 9].

Definition 1: Let $(X, \mathcal{B})$ be a separable measurable space with a distinguished point ω and a measurable map $\Phi : R_+ \times X \to X$ having the following properties:

(S$_1$) for any $x \in X$ there exists an element $\rho(x) \in [0,+\infty]$ such that $\Phi(t,x) \neq \omega$ for all $t \in [0, \rho(x)[$ and $\Phi(t,x) = \omega$ for all $t \geq \rho(x)$,

(S$_2$) for any $s, t \in R_+$ and any $x \in X$ we have

$$\Phi(s, \Phi(t,x)) = \Phi(s+t, x),$$

(S$_3$) $\Phi(0,x) = x$, for all $x \in X$,

(S$_4$) if $\Phi(t,x) = \Phi(t,y)$, for all $t > 0$, then $x = y$.

The collection $(X, \mathcal{B}, \Phi, \omega)$ is called semidynamical system with a coffin state ω.

Set $X_0 = X \backslash \{\omega\}$. For any $x \in X_0$ we denote by Γ_x the trajectory of x, i.e.:

$\Gamma_x = \{\Phi(t,x), t \in [0, \rho(x)[\}$ and we define the function Φ_x on $[0, \rho(x)[$ by $\Phi_x(t) = \Phi(t,x)$. So for any $x, y \in X_0$ we put $x \leq_\Phi y$ if $y \in \Gamma_x$. A maximal trajectory is a totally ordered subset Γ of X_0 with respect to the above order, such that there is no $x_0 \in X_0 \backslash \Gamma$ which is a minorant of Γ and such for any $x \in \Gamma$, we have $\Gamma_x \subset \Gamma$.

In what follows, we shall suppose that $(X, \mathcal{B}, \Phi, \omega)$ is a transient semidynamical system [1, 3]. In [1] we have associated a proper and submarkovian resolvent $\mathbf{V} = (V_\alpha)_{\alpha \geq 0}$ of kernels on the measurable space $(X_0, \mathcal{B}_0)$, defined by :

$$V_\alpha f(x) = \int_0^{\rho(x)} e^{-\alpha t} f(\Phi(t,x))dt, \forall x \in X_0, \forall \alpha \in R_+,$$

where, $\mathcal{B}_0 = \{U \subset \mathcal{B}; U \subset X_0\}$.

The family $\mathbf{V}$ is the resolvent associated to the deterministic semigroup $\mathbf{H} = (H_t)_{t \geq 0}$ introduced in [10, 11].

It is proved that the map Φ_x is a measurable isomorphism between $[0, \rho(x)[$ and Γ_x endowed with trace measurable structures.

Let Λ be the Lebesgue measure associated with the semidynamical system $(X, \mathrm{B}, \Phi, \omega)$ [9] given by $\Lambda(A) = \lambda(\Phi_x^{-1}(A))$ for any $x \in X_0, A \in \mathrm{B}_0$ and $A \subset \Gamma_x$, where λ denotes the Lebesgue measure on R. We recall [5] that in the same way Λ can be defined on the σ-algebra $\mathrm{B}_0(\Lambda)$ the set of all subsets A of X_0 such that $A \cap M \in \mathrm{B}_0$ for any countable union M of trajectories of X_0. Set B_0^{σ} the collection of all countable union of trajectories of X_0. We denote by $\mathrm{F}(X_0, \Lambda)$, the set of all nonnegative $\mathrm{B}_0(\Lambda)$-measurable functions on X_0. One can show that the resolvent family $\mathbf{V}$ may be considered on the measurable space ($X_0, \mathrm{B}_0(\Lambda)$) by setting also :

$$V_\alpha f(x) = \int_0^{\rho(x)} e^{-\alpha t} f(\Phi(t,x))dt, \forall x \in X_0, \forall \alpha \in R_+.$$

We consider also the arrival time function $\Psi : X_0 \times X_0 \to R_+$, given [2] by :

$\Psi(x,y) = t$ if $\Phi(t,x) = y, t \in [0, \rho(x)[$ and

$\Psi(x,y) = +\infty$ if not.

It is shown that the arrival time function Ψ is measurable if we endow $X_0 \times X_0$ with the product measurable structure of the σ-algebra $\mathrm{B}_0(\Lambda)$ [5, 9]. Using the Lebesgue measure Λ and the function Ψ, we associate to the semidynamical system a dual resolvent $\mathbf{V}^* = (V_\alpha^*)_{\alpha \geq 0}$ of kernels on the measurable space ($X_0, \mathrm{B}_0(\Lambda)$) with respect to Λ, [5]. The above resolvent is given on $\mathrm{F}(X_0, \Lambda)$ by:

$$V_\alpha^* f(x) = \int e^{-\alpha\Psi(y,x)} G(y,x) f(y) d\Lambda(y), \forall x \in X_0, \forall \alpha \in R_+,$$

where, $G(y,x) = 1$, if $y \leq_\Phi x$ and $G(y,x) = 0$, if not, define the Green function associated to $(X, \mathrm{B}, \Phi, \omega)$.

For each $x \in X_0$, let us denote by:

$$v_x = \{v \subset X_0 : \exists \alpha \in]0, \rho(x)[/ \Phi(t,x) \in v, \forall t \in [0, \alpha[\}$$

and let τ_Φ be the topology for which v_x generates all the neighborhoods of x [1]. τ_Φ is namely the fine topology associated with $(X, \mathrm{B}, \Phi, \omega)$.

Also denote by τ_Φ^0 the inherent topology [1] associated with $(X, \mathrm{B}, \Phi, \omega)$ which is characterized as being the set of all subset D of X_0 having the following property:

$$\forall x \in X_0, \forall t_0 \in [0, \rho(x)[: \Phi(t_0, x) \in D \Rightarrow$$
$$\Phi(t,x) \in D, \forall t \in]t_0 - \varepsilon, t_0 + \varepsilon[\cap [0, \rho(x)[,$$

for some $\varepsilon > 0$.

Theorem 1 : The set $\mathrm{E}(\Lambda)$ of the $\mathbf{V}$-excessive functions on ($X_0, \mathrm{B}_0(\Lambda)$) is identical to the set of all positive decreasing functions on X_0 with respect to the order '$\leq_\Phi$', continuous with respect to the fine topology τ_Φ and finite at the points $x \in X_0$ which are not minimal with respect to the same order [12]. Thus, the following result holds:

Proposition 1: Any function $f \in \mathrm{E}(\Lambda)$ is lower semicontinuous with respect to τ_Φ^0.

Proof : Since $\mathbf{V}$ is a proper submarkovian resolvent on ($X_0, \mathrm{B}_0(\Lambda)$), then by Hunt's approximation theorem, [13] page 23, there exists a sequence $(f_n)_n$ in $\mathrm{F}(X_0, \Lambda)$ such that $\sup_n V_0 f_n = f$. Since $V_0 f_n$ is continuous with respect to τ_Φ^0 [1], then f is lower semicontinuous with respect to τ_Φ^0.

Next, we shall prove the following Theorem which will be needed later.

Theorem 2: The following properties hold:

(i) Every τ_Φ-open set is $\mathrm{B}_0(\Lambda)$-measurable,

(ii) Every monotonous function f with respect to '$\leq_\Phi$' is $\mathrm{B}_0(\Lambda)$-measurable.

Proof

1. Let $O \in \tau_\Phi$. Using a result in [1], $\Gamma_x \in \tau_\Phi$, we get that $O \cap \Gamma_x \in \tau_\Phi$ which means that $\Phi_x^{-1}(O \cap \Gamma_x)$ is an open set with respect to the fine trace topology on $[0, \rho(x)]$. Thus, it is measurable with respect to the trace Borel σ-algebra. Using the fact that Φ_x is a measurable isomorphism, we get that $O \cap \Gamma_x \in \mathrm{B}_0$ and therefore $O \cap \Gamma_x \in \mathrm{B}_0(\Lambda)$.
2. The function g defined by $g(t) = f \circ \Phi(t,x) = f \circ \Phi_x(t)$ is monotonous on $[0, \rho(x)]$ which is measurable with respect to trace Borel σ-algebra on $[0, \rho(x)]$. Using the fact that Φ_x is a measurable isomorphism, we get that

$f = g \circ \Phi_x^{-1}$ is B_0-measurable and then f is $B_0(\Lambda)$-measurable.
In the sequel, for any subset A of X_0 we put $A^c := X_0 \setminus A$.

On the Measurability of the Hitting Time: Blumenthal and Getoor [7] proved the measurability of the hitting time for any Borel measurable subset A in the case of locally compact separable metric space.

Proposition 2: Let T be a positive measurable function on X with respect to the σ-algebra B (respectively $B(\Lambda)$), then the map Φ_T, given by $\Phi_T(x) = \Phi(T(x), x)$ if $T(x) < \rho(x)$ and $\Phi_T(x) = \omega$ if not, is measurable with respect to B (respectively $B(\Lambda)$).

Proof: The map U given on X by $U(x) = (T(x),x)$ is measurable on X with respect B (respectively $B(\Lambda)$) and the product measurable structure. Then $\Phi_T = \Phi \circ U$ is also measurable on X with respect to B (respectively $B(\Lambda)$).

Notation: For any positive function f defined on X_0, we put:
$f(\Phi_T)(x) = f(\Phi_T(x))$ if $T(x) < \rho(x)$ and $f(\Phi_T)(x) = 0$ if not.

Definition 2: Let $A \in B(\Lambda)$ and for all $x \in X$ we put
$D_A(x) = \inf\{t \geq 0 : \Phi(t,x) \in A\}$ if there exists and $D_A(x) = +\infty$ if not.
Also we put
$T_A(x) = \inf\{t > 0 : \Phi(t,x) \in A\}$ if there exists and $T_A(x) = +\infty$ if not.
The function D_A (respectively T_A) is called the first entry time (respectively the first hitting time) of A.
Note that, for any $A \in B_0(\Lambda)$, if $D_A(x) \geq \rho(x)$ (respectively $T_A(x) \geq \rho(x)$), then
$D_A(x) = T_A(x) = +\infty$.

Example: The life time ρ is the first entry time or also the first hitting time in $\{\omega\}$.
Note that $D_A(x) = 0$, for any $x \in A$.

Definition 3: Let $A \in B(\Lambda)$. A point x is called regular for A if $T_A(x) = 0$ and it is irregular if $T_A(x) > 0$. We denote by A^r the set of all regular points of A, i.e. $A^r = [T_A = 0]$.

Remark 1: Let $A \in B_0(\Lambda)$ and denote by $\mathring{A}$ the fine interior of A and by $\bar{A}$ the set of all adherent points with respect to the fine topology τ_Φ. Then $\mathring{A} \subset A^r \subset \bar{A}$.

Proposition 3: Let $A \in B$ (respectively $B(\Lambda)$). The following properties hold:
(i) $s + D_A(\Phi(s,x)) = \inf\{t \geq s : \Phi(t,x) \in A\}$,
(ii) $s + T_A(\Phi(s,x)) = \inf\{t > s : \Phi(t,x) \in A\}$,
(iii) $t + D_A(\Phi(t,x)) = D_A(x)$ on the set $[D_A \geq t]$,
(iv) $t + T_A(\Phi(t,x)) = T_A(x)$ on the set $[T_A > t]$,
(v) $D_A \leq T_A$ and $D_A = T_A$ if $x \notin A$.

Corollary 1: Let $A \in B_0(\Lambda)$. The following properties hold:
(i) $t + D_A(\Phi(t,x)) \geq D_A(x)$,
(ii) $t + T_A(\Phi(t,x)) \geq T_A(x)$,
(iii) D_A is continuous with respect to the fine topology τ_Φ on each point of the set $[D_A > 0]$.
(iv) T_A is continuous with respect to the fine topology τ_Φ on each point of the set $[T_A > 0]$.

Proof
(i) and (ii) are obvious by using (i) and (ii) in the last Proposition.
(iii)Let $x \in X_0$ be such that $D_A(x) > 0$. Then, there exists $0 < \alpha < D_A(x)$ and therefore $D_A(x) > t$ for any $t \in [0,\alpha]$. Using the last Proposition, we conclude that $t + D_A(\Phi(t,x)) = D_A(x)$, for any $t \in [0,\alpha]$ and that $\lim_{t \to 0} D_A(\Phi(t,x)) = D_A(x)$ i.e. D_A is continuous with respect to τ_Φ on the set $[D_A > 0]$.
In the same way we prove (iv).

Proposition 4: Let A be an open subset of X_0 with respect to τ_Φ. Then $D_A(x) = T_A(x)$ on X_0 [7]. Moreover D_A is continuous with respect to the fine topology τ_Φ. Particularly, D_A is measurable with respect $B_0(\Lambda)$.

Proof: Since $D_A \leq T_A$, then $D_A = 0$ if $T_A = 0$. Now, if $T_A(x) > 0$ for some $x \in X_0$, then $x \notin A$ and $D_A(x) = T_A(x)$. In fact, if $x \in A$, there exists $\varepsilon > 0$ such that $\Phi(t,x) \in A$, for all $t \in [0,\varepsilon[$ and therefore $T_A(x) = 0$.

Now, let $x \in X_0$ be such that $D_A(x) = 0$ i.e. $T_A(x) = 0$ and there exists a non increasing sequence $(t_n)_n \in (R_+^*)^N$ such that $\Phi(t_n, x) \in A$ and $\lim_{n \to +\infty} t_n = 0$. So, for any $\varepsilon > 0$, there exists n_0 such that $t_{n_0} < \varepsilon$ and therefore $D_A(\Phi(t,x)) \le t_{n_0} - t < \varepsilon$, $\forall t \in [0, t_{n_0}[$. So that D_A is continuous at x with respect to the fine topology τ_Φ. The proof then holds by Corollary 1.

Proposition 5: Let $A, B \in B_0(\Lambda)$. Then, the following assertions hold [7]:

(i) $A \subset B \Rightarrow D_B \le D_A$ and $T_B \le T_A$,

(ii) $D_{A \cup B} = \inf\{D_A, D_B\}$ and $T_{A \cup B} = \inf\{T_A, T_B\}$,

(iii) $\sup\{D_A, D_B\} \le D_{A \cap B}$ and $\sup\{T_A, T_B\} \le T_{A \cap B}$,

(iv) for any increasing sequence $(A_n)_n$ of measurable subsets of X_0 such that $A = \bigcap_n A_n$, we have $D_A = \inf_n D_{A_n}$ and $T_A = \inf_n T_{A_n}$.

In the sequel set $x < y$ for any $x, y \in X_0$ such that $x \le_\Phi y$ with $x \ne y$ and set:

1. $[x,y] = \{z \in X_0 : x \le_\Phi z \le_\Phi y\}$,
2. $[x,y[= \{z \in X_0 : x \le_\Phi z < y\}$,
3. $]x,y[= \{z \in X_0 : x < z < y\}$,
4. $]x,y] = \{z \in X_0 : x < z \le_\Phi y\}$.

Proposition 6 : Let A be a closed subset of X_0 with respect to τ_Φ. Then, T_A is continuous with respect to τ_Φ.

Proof: Let $x_0 \in X_0$ be such that $T_A(x_0) < +\infty$ (obviously that $T_A(x_0) = +\infty \Rightarrow T_A(x) = +\infty, \forall x \in \Gamma_{x_0}$).

First Case: If $x_0 \in X_0 \setminus A$, then there exists $\varepsilon > 0$ such that $[x_0, \Phi(\varepsilon, x_0)[\subset X_0 \setminus A$ and so $T_A(x_0) > \Psi(x_0, x)$, $\forall x \in [x_0, \Phi(\varepsilon, x_0)[$. Then, $T_A(x_0) = T_A(x) + \Psi(x_0, x)$ and $0 \le T_A(x_0) - T_A(x) < \varepsilon$, $\forall x \in [x_0, \Phi(\varepsilon, x_0)[$.

Second Case: If $x_0 \in \overset{0}{A}$, then there exists $\varepsilon > 0$ such that $[x_0, \Phi(\varepsilon, x_0)[\subset A$ and so $T_A(x) = 0$, $\forall x \in [x_0, \Phi(\varepsilon, x_0)[$.

Third Case: If $x_0 \in \partial A = A \setminus \overset{0}{A}$, then for any $\varepsilon > 0$ there exists $b \in [x_0, \Phi(\varepsilon, x_0)[\cap (X_0 \setminus A)$. If there exists $\varepsilon > 0$ such that $]x_0, \Phi(\varepsilon, x_0)[\cap A = \emptyset$, so $T_A(x_0) > \Psi(x_0, x)$ and $T_A(x_0) = T_A(x) + \Psi(x_0, x)$, $\forall x \in]x_0, \Phi(\varepsilon, x_0)[$. Then, for any $x \in]x_0, \Phi(\varepsilon, x_0)[$, we have $0 \le T_A(x_0) - T_A(x) < \varepsilon$.

Finally, suppose that for any $\varepsilon > 0$ there exists $a \in [x_0, \Phi(\varepsilon, x_0)[\cap A$ and therefore for $\alpha = \Psi(x_0, a)$, we have $T_A(x) \le \Psi(x_0, a) < \varepsilon$, $\forall x \in [x_0, \Phi(\alpha, x_0)[$.

In particular, $T_A(x_0) = 0$ and for any $x \in [x_0, \Phi(\alpha, x_0)[$, $0 \le T_A(x) - T_A(x_0) < \varepsilon$.

In the different cases cited above, we conclude that T is continuous at x_0 with respect to τ_Φ.

Remark 2: In the proof given above we can deduce that T_A is continuous at x_0 in the first and the third case by using Corollary 1.

Theorem 3: For any $A \in B_0(\Lambda)$, we have the following assertions:

(i) $D_A = D_{\bar{A}}$ and $T_A = T_{\bar{A}}$,

(ii) D_A (T_A resp.) is lower semicontinuous (continuous resp.) with respect to the fine topology τ_Φ.

(iii) D_A and T_A are measurable on X_0 with respect to $B_0(\Lambda)$.

Proof

(i) Obviously, $D_{\bar{A}} \le D_A$ and $T_{\bar{A}} \le T_A$ (Proposition 5). Next, let $x \in X_0$ be such that $D_{\bar{A}}(x) < +\infty$ ($T_{\bar{A}}(x) < +\infty$ resp.) and let $t \ge 0$ ($t > 0$ resp.) be such that $\Phi(t,x) \in \bar{A}$. By using Proposition 3 in [1] we deduce that for any $n \in N^*$ there exists $t_n \in [0, \frac{1}{n}[$ such that $\Phi(t + t_n, x) \in A$ and therefore

$D_A(x) \le t + t_n \le t + \frac{1}{n}$ ($T_A(x) \le t + t_n \le t + \frac{1}{n}$ resp.).

Hence, $D_A(x) \le D_{\bar{A}}(x) + \frac{1}{n}$

($T_A(x) \leq T_{\bar{A}}(x) + \frac{1}{n}$ resp.), for any $n \in N^*$.
Consequently $D_A(x) \leq D_{\bar{A}}(x)$ ($T_A(x) \leq T_{\bar{A}}(x)$ resp.). Note that if $D_{\bar{A}}(x) = +\infty$ ($T_{\bar{A}}(x) = +\infty$ resp.), then $D_A(x) = +\infty$ ($T_A(x) = +\infty$ resp.).

(ii) Let $\alpha \geq 0$ and let $(x_n)_n$ be a sequence in $[D_A(x) \leq \alpha]$ which is converges to $x_0 \in X_0$ with respect to τ_Φ. If $D_A(x_0) = 0$, then $x_0 \in [D_A(x) \leq \alpha]$. If $D_A(x_0) > 0$, then by Corollary1 D_A is continuous at x_0 w.r. to τ_Φ. So that $(D_A(x_n))_n$ converges to $D_A(x_0)$ and therefore $x_0 \in [D_A(x) \leq \alpha]$. By Proposition6 we get that T_A is continuous with respect to τ_Φ.

(iii) It is obvious by (ii).

Proposition 7: Let $A \in B_0(\Lambda)$. So, we have $D_A = \inf\{D_{A\cap M}, M \in B_0(\Lambda)\}$ and $T_A = \inf\{T_{A\cap M}, M \in B_0(\Lambda)\}$.

Proof: Obviously, for any $M \in B_0(\Lambda)$, we have $D_A \leq D_{A\cap M}$ and $T_A \leq T_{A\cap M}$.
Next, we denote by $B_0{}^\sigma$ the set of all countable unions of trajectories of X_0 and let $x_0 \in X_0$ such that $D_A(x_0) < \alpha$ (respectively $T_A(x_0) < \alpha$) for some real number α. Then, there exists $t \in [0, \alpha]$ (respectively $t \in]0, \alpha[$) such that $\Phi(t, x_0) \in A$. But $A = \bigcup_M A \cap M$, where M rains the $\sigma -$algebra $B_0{}^\sigma$, so there exists $M_0 \in B_0{}^\sigma$ such that $\Phi(t, x_0) \in A \cap M_0$ so that $D_{A\cap M_0}(x_0) < \alpha$ (respectively $T_{A\cap M_0}(x_0) < \alpha$).
Then, we have

$D_A(x_0) = \inf\{D_{A\cap M}(x_0), M \in B_0(\Lambda)\}$ and

$T_A(x_0) = \inf\{T_{A\cap M}(x_0), M \in B_0(\Lambda)\}$.

Remark 3: Notice that for any $A \in \mathbf{B_0}(\Lambda)$ and any $x_0 \in X_0$ we have $x \leq_\Phi \Phi_{D_A}(x)$ and $x \leq_\Phi \Phi_{T_A}(x)$.

Notation: For any $x,y \in X_0$ we write $x < y$ if $x \leq_\Phi y$ and $x \neq y$.

Proposition 8: Let $A \in B_0(\Lambda)$. Then, the following properties hold:

(i) The maps Φ_{D_A} and Φ_{T_A} are increasing,in particular Φ_{D_A} and Φ_{T_A} are measurable on X_0 with respect to $B_0(\Lambda)$.

(ii) $\forall x_1, x_2 \in X_0$, $x_1 < x_2$ we have
$\Phi_{D_A}(x_1) = \Phi_{D_A}(x_2) \Leftrightarrow [x_1,x_2[\cap A = \emptyset$ (respectively
$\Phi_{T_A}(x_1) = \Phi_{T_A}(x_2) \Rightarrow [x_1,x_2[\cap \overset{0}{A} = \emptyset$),

(iii) If A is dense in X_0 with respect to τ_Φ, then $D_A = T_A = 0$ and for any $x_1,x_2 \in X_0$ we have $x_1 < x_2 \Rightarrow \Phi_{D_A}(x_1) = \Phi_{D_A}(x_2)$.

Proof

(i) Let $x_1, x_2 \in X_0$, $x_1 < x_2$. Then we obtain
$D_A(x_1) \leq D_A(x_2) + \Psi(x_1, x_2)$ and
$T_A(x_1) \leq T_A(x_2) + \Psi(x_1, x_2)$ (Corollary 1).
Also we deduce that $\Phi_{D_A}(x_1) \leq_\Phi \Phi_{D_A}(x_2)$ and $\Phi_{T_A}(x_1) \leq_\Phi \Phi_{T_A}(x_2)$.

(ii) Let $x_1, x_2 \in X_0$, $x_1 < x_2$ be such that $[x_1,x_2[\cap A \neq \emptyset$ and let $a \in [x_1,x_2[\cap A$. Then, we have
$D_A(x_1) \leq \Psi(x_1, a) < \Psi(x_1, x_2)$.
Therefore we get $\Phi_{D_A}(x_1) < \Phi(\Psi(x_1, x_2), x_1)$,
i.e. $\Phi_{D_A}(x_1) < x_2 \leq_\Phi \Phi_{D_A}(x_2)$.
Conversely, if $[x_1,x_2[\cap A = \emptyset$, we get $D_A(x_1) = D_A(x_2) = +\infty$ or there exists $a_0 \in A$ such that $x_1 < x_2 < a_0$ and therefore
$\Phi_{D_A}(x_1) = \Phi_{D_A}(x_2) = \wedge\{a \in A : x_2 < a\}$, the infinimum with respect to the associated order.

Now let $a \in [x_1,x_2[\cap \overset{0}{A}$, so there exists $\varepsilon > 0$ such that $[a, \Phi(\varepsilon, a)[\subset [x_1,x_2[\cap \overset{0}{A}$. We consider $a_1 \in]a, \Phi(\varepsilon, a)[$ and therefore
$T_A(x_1) \leq \Psi(x_1, a_1) < \Psi(x_1, x_2)$.
Then, $\Phi_{T_A}(x_1) < x_2 \leq_\Phi \Phi_{T_A}(x_2)$.

(iii) Suppose that A is a dense subset of X_0 with respect to τ_Φ.
So for any $x \in X_0$ and any $n \in N^* \cap \left]\frac{1}{\rho(x)}, +\infty\right[$,
there exists $x_n \in A \cap [x, \Phi(\frac{1}{n}, x)[$ and therefore
$D_A(x) \leq T_A(x) = 0$.

Proposition 9: Let $A \in B_0(\Lambda)$. Then for any $x \in X_0$ such that $D_A(x) < +\infty$ (respectively $T_A(x) < +\infty$) we have $\Phi_{D_A}(x) \in \bar{A}$ (respectively $\Phi_{T_A}(x) \in \bar{A}$).

Proof: For any $x \in X_0$ such that $D_A(x) < +\infty$ (respectively $T_A(x) < +\infty$) there exists a decreasing sequence $(t_n)_n$ of positive real numbers such that $D_A(x) = \lim_n t_n$ (respectively $T_A(x) = \lim_n t_n$) and for any n, $\Phi(t_n, x) \in A$. Since $t \mapsto \Phi(t, x)$ is a right continuous map with respect to τ_Φ, then we have $\Phi_{D_A}(x) \in \bar{A}$ (respectively $\Phi_{T_A}(x) \in \bar{A}$).

Corollary 2: Let A be a closed subset of X_0 with respect to τ_Φ. Then, for any $x \in X_0$ such that $D_A(x) < +\infty$ (respectively $T_A(x) < +\infty$) we have $\Phi_{D_A}(x) \in A$ (respectively $\Phi_{T_A}(x) \in A$).

Remark 4: Let $A \in B_0(\Lambda)$ and $x \in \overset{0}{A}$. Then $T_A(x) = 0$.

Proof : Let $x \in \overset{0}{A}$, then there exists $\varepsilon > 0$ such that $[x, \Phi(\varepsilon, x)[\ \subset A$ and therefore $T_A(x) = 0$.

A Hunt Theorem for Semidynamical Systems: In this section $\mathbf{S}$ (respectively $\mathbf{E}$) will simply denote the set of all supermedian (respectively excessive) functions with respect to the extension resolvent $\mathbf{V}$.

Definition 4: Let $A \in B_0(\Lambda)$ and let $s \in \mathbf{E}$. The map ${}^S R_s^A$ (respectively R_s^A) given on X_0 by, the pointwise infinimum [13],
${}^S R_s^A := \inf\{t \in \mathbf{S} : t \geq s \text{ on } A\}$ (respectively $R_s^A := \inf\{t \in \mathbf{E} : t \geq s \text{ on } A\}$) is called the reduite of s on A with respect to $\mathbf{S}$ (respectively $\mathbf{E}$).

Definition 5 : For any $A \in B_0(\Lambda)$ and any $s \in \mathbf{E}$ the map B_s^A given on X_0 by [13]:
$B_s^A := \wedge \inf\{t \in \mathbf{E} : t \geq s \text{ on A}\}$,
where, the infinimum is considered in $\mathbf{E}$, is called the balayage of s on A.

Theorem 4: Let $A \in B_0(\Lambda)$ and let $s \in \mathbf{E}$. Then, we have $B_s^A = \hat{R}_s^A := \sup_{\alpha > 0} \alpha V_\alpha(R_s^A)$, the excessive regularization. In particular $B_s^A = R_s^A$ $\Lambda -$a.e.

Proof: Let $(t_i)_{i \in I}$ be a family of all the elements of $\mathbf{E}$ such that $t_i \geq s$ on A. Then, we have $R_s^A = \inf_{i \in I} t_i$ (Note that R_s^A is measurable with respect to $B_0(\Lambda)$ by Theorem 2) which is supermedian with respect to $\mathbf{V}$ and $\hat{R}_s^A$ is in $\mathbf{E}$ and so $\hat{R}_s^A = \wedge_{i \in I} t_i = B_s^A$, where, $\wedge_{i \in I} t_i$ is the greatest lower bound in $\mathbf{E}$.
The following result is due to Mokobodzki. For the proof one can see [13], pages 9 and 13.

Proposition 10 : For any $A \in B_0(\Lambda)$ and any $s \in \mathbf{S}$, the function ${}^S R_s^A$ is supermedian with respect to $\mathbf{V}$.
For the following result also see [13] .

Proposition 11: For any $A \in B_0(\Lambda)$ and any $s \in \mathbf{E}$, we have $R_s^A = R_s^{\bar{A}}$.

Proof: It is obvious that $R_s^A \leq R_s^{\bar{A}}$. Now, let $t \in \mathbf{E}$ such that $t \geq s$ on A. since s and t are continuous with respect to τ_Φ, we get $t \geq s$ on $\bar{A}$ and therefore $t \geq s$ on $\bar{A}$ which implies that $R_s^A \geq R_s^{\bar{A}}$.

Corollary 3: For any $A \in B_0(\Lambda)$ and any $s \in \mathbf{E}$, we have $s(\Phi_{T_A}) = R_s^A$.

Proof: Let $x \in X_0$ be such that $T_A(x) < +\infty$ and let $u \in \mathbf{E}$ be such that $u \geq s$ on $\bar{A}$. Since by Proposition9, $\Phi_{T_A}(x) \in \bar{A}$, then
$u(x) \geq u(\Phi_{T_A}(x)) \geq s(\Phi_{T_A}(x))$ and hence
$R_s^{\bar{A}}(x) \geq s(\Phi_{T_A}(x))$, which yields by Proposition11 that $R_s^A \geq s(\Phi_{T_A})$.
Note that if $T_A(x) = +\infty$, then $s(\Phi_{T_A}(x)) = 0$.

Theorem 5: Let A be a closed subset of X_0 with respect to τ_Φ and let $s \in \mathbf{S}$. Then, we have
${}^S R_s^A = s(\Phi_{D_A})$.

Proof : Since $s(\Phi_{D_A})$ is a decreasing function (Proposition 8), then $s(\Phi_{D_A}) \in \mathbf{S}$. Moreover for any

$x \in A$ we have $s(\Phi_{D_A}(x)) = s(x)$. Now, let $t \in \mathbf{S}$ such that $t \geq s$ on A. Since A is closed with respect to τ_Φ, then $\Phi_{D_A}(x) \in A$ for any $x \in X_0$ such that $D_A(x) < +\infty$ and therefore $t(x) \geq t(\Phi_{D_A}(x)) \geq s(\Phi_{D_A}(x))$.
Now, if $D_A(x) = +\infty$, then $\Phi_{D_A}(x) = \omega$ and $t(x) \geq s(\Phi_{D_A}(x)) = 0$.

Theorem 6: Let A be a fine open subset of X_0 and let $s \in \mathbf{E}$. Then, we have $B_s^A = R_s^A = s(\Phi_{D_A})$.

Proof: For any $x \in X_0$ and $t \in [0, \rho(x)[$, we have $\Phi_{D_A}(\Phi(t,x)) = \Phi(D_A(\Phi(t,x)), \Phi(t,x)) = \Phi(t + D_A(\Phi(t,x)), x)$.
But D_A is continuous with respect to τ_Φ (Proposition 4), so $\lim_{t \to 0^+}(t + D_A(\Phi(t,x))) = D_A(x)$ and therefore $s(\Phi_{D_A})$ is continuous with respect to τ_Φ. Since $s(\Phi_{D_A})$ is a decreasing function and continuous with respect to the fine topology τ_Φ, then $s(\Phi_{D_A}) \in \mathbf{E}$. Now, let $t \in \mathbf{E}$ such that $t \geq s$ on A. Since s and t are continuous with respect to the fine topology τ_Φ, so $t \geq s$ on $\bar{A}$. But for any $x \in X_0$ such that $D_A(x) < +\infty$, we have that $\Phi_{D_A}(x) \in \bar{A}$ and therefore $t(x) \geq t(\Phi_{D_A}(x)) \geq s(\Phi_{D_A}(x))$.
Now, if $D_A(x) = +\infty$, then $\Phi_{D_A}(x) = \omega$ and $t(x) \geq s(\Phi_{D_A}(x)) = 0$.

Remark 5: For the case of continuous semidynamical system on a locally compact space with countable base in [11] is proved the above statement for any complement of compact set.

Theorem 7 : Let $A \in \mathrm{B}_0(\Lambda)$ and $s \in \mathbf{E}$. Then, [13, 14] we have $R_s^A = \inf\{R_s^G : G \in \tau_\Phi, A \subset G\}$.

Proof: Obviously, $R_s^A \leq \inf\{R_s^G : G \in \tau_\Phi, A \subset G\}$. Now, let $t \in \mathbf{E}$ be such that $t \geq s$ on A. Then, for every $\varepsilon > 0$ the subset $G_\varepsilon = [t + \varepsilon > s]$ is a fine open subset which contains A. Since $t + \varepsilon > s$ on G_ε, it follows that $t + \varepsilon \geq R_s^{G_\varepsilon} \geq \inf\{R_s^G : G \in \tau_\Phi, A \subset G\}$ and therefore $t \geq \inf\{R_s^G : G \in \tau_\Phi, A \subset G\}$.
Consequently $R_s^A \geq \inf\{R_s^G : G \in \tau_\Phi, A \subset G\}$.

In the sequel we formulate and we give a direct proof of Hunt's fundamental Theorem which is proved in [6-8]. In our case, we don't assume that the state space X is a locally compact metric space with countable base.

Theorem 8: Let $A \in \mathrm{B}_0(\Lambda)$ and $s \in \mathbf{E}$. Then $s(\Phi_{T_A}) \leq R_s^A$ on X_0 and $s(\Phi_{T_A}) = R_s^A$ except on $A \cap (A^r)^c$.

Proof: By Corollary3, we have that $s(\Phi_{T_A}) \leq R_s^A$ for any $A \in \mathrm{B}_0(\Lambda)$. Next, let $x \in A^r$ (i.e. $T_A(x) = 0$). Since $R_s^A \leq s$, then $s(\Phi_{T_A}(x)) \leq R_s^A(x) \leq s(x) = s(\Phi_{T_A}(x))$. Hence $s(\Phi_{T_A}) \leq R_s^A$ on A^r.
Now, let $x \in A^c$ be such that $T_A(x) = +\infty$. It is obvious that $T_A = D_A$ on A^c (see Proposition3) and therefore $D_A(x) = +\infty$. Hence, we deduce that $A \subset \Gamma_x^c$. Since by Theorem 1, $1_{\Gamma_x^c} \in \mathbf{E}$, then $s1_{\Gamma_x^c} \in \mathbf{E}$.
On the other hand, using the fact that $s1_{\Gamma_x^c} = s$ on A, we obtain $R_s^A(x) = 0 = s(\Phi_{T_A}(x))$.
Next, let $x \in A^c \cap (A^r)^c$ be such that $T_A(x) < +\infty$, then the ordered interval $U_x = [x, \Phi_{T_A}(x)[$ is a fine open and also a fine closed non empty subset of X_0 (Proposition 4) in [1] and $U_x \subset A^c$. Let us set $s_x = s(\Phi_{T_A})$ on U_x and $s_x = s$ on $(U_x)^c$.
Obviously, s_x is continuous on U_x with respect to τ_Φ. Also, $s_x = s$ on $(U_x)^c$ on the fine open subset $(U_x)^c$ with respect to τ_Φ. Since $X_0 = U_x \cup (U_x)^c$ and $U_x \cap (U_x)^c = \emptyset$, we get that s_x is τ_Φ-continuous. Using the fact that s_x is decreasing on X_0, we obtain by Theorem1 that $s_x \in \mathbf{E}$. But the fact that $A \subset U_x^c$ gives us that $s_x = s$ on A and therefore $R_s^A(x) \leq s_x$. Particularly, $R_s^A(x) \leq s(\Phi_{T_A}(x))$. Hence $R_s^A(x) = s(\Phi_{T_A}(x))$ and consequently $R_s^A = s(\Phi_{T_A})$ on $X_0 \setminus (A \cap (A^r)^c)$.

Remark 6: In [1] it is proved the above statement for any $A \in \mathrm{B}_0$ in the case where $s = 1$ on X_0.

Corollary 4: Let A be a subset of X_0 which is closed with respect to τ_Φ and such that $\Lambda\,(A \backslash A^r) = 0$ and let $s \in \mathbf{E}$. Then, we have $B_s^A = s(\Phi_{T_A})$.

Proof: Since $s(\Phi_{T_A})$ is a decreasing function and continuous with respect to τ_Φ, then, by Proposition 6 and Theorem 1, $s(\Phi_{T_A}) \in \mathbf{E}$ [1, 12]. On the other hand, by Theorem 4:

$B_s^A = \hat{R}_s^A = R_s^A$, Λ-a.e. and $R_s^A = s(\Phi_{T_A})$ except on $A \backslash A^r$, which is Λ- negligible. Then, $B_s^A = s(\Phi_{T_A})$ Λ-a.e. Since B_s^A and $s(\Phi_{T_A}) \in \mathbf{E}$, we get [5]

$B_s^A = s(\Phi_{T_A})$.

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