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Measures on the Quotient Spaces of the Integers

Norris Sookoo and Peter Chami
[1]Department of Mathematics and Computer Science,
University of the West Indies, St. Augustine,
Trinidad, W.I.

Abstract: SK-partitions were introduced by Sharma and Kaushik, who defined distances (metrics) between vectors in terms of partitions of the alphabet set F_q, the set of the ring of integers modulo q. These distances were applied in Coding Theory. The research examined algebraic and topological aspects of SK-partitions and related sets of measures. A lattice of SK-partitions was introduced and shown to have distributive sub lattices. Generator sets of the lattice were obtained and the structure of its ideals and filters examined. Corresponding results for lattices of measures were presented.

Keywords: Measure, Lattice, Metric, Ideal, Generator set

INTRODUCTION

Measures on the quotient spaces of the integers were studied by Niederreiter and Sookoo [9] in connection with the uniform distribution of sequences. Systems of measures were studied by Maharan [8], who considered a family of measures with orthogonality properties and also by Schmidt[10], who proved that a certain ordered Banach space of vector measures is a Banach lattice. We examined the algebraic aspects of systems of measures defined on quotient spaces of the integers modulo q for different natural numbers q. These measures were defined in terms of SK-partitions of the ring of integers modulo q. Studies in Coding Theory involving SK-Partitions were carried out by Kaushik[2,3,4,5,6,7], Sharma and Dial [11] and Sharma and Kaushik[12,13,14]. From results obtained we deduced comparable results for systems of measures.

DEFINITIONS AND NOTATIONS

Notation: Let_F_q = {0, 1,…., q-1} be the ring of integers modulo q, $q \in \{2,3,\dots\}$.

Definition: Given F_q, $q \geq 2$, a partition P =\{ $B_0, B_1, \dots, B_{m-1}$ \} of F_q is called an <u>SK-partition</u> if

1. $B_0 = \{0\}$, and q-$a \in B_i$ if $a \in B_i$, i = 1,2, …., m-1
2. If $a \in B_i$ and $b \in B_j$; i ,j=0,1,…,m-1, and if j precedes i in the order of the partition P, written as i>j, then min{a, q-a} > min{b, q-b}.
3. If i>j ($i, j \in \{0,1,\dots,m-1\}$) and $i \neq m-1$, then
   $|B_i| \geq |B_j|$ and $|B_{m-1}| \geq \frac{1}{2}|B_{m-2}|$,
   where $|B_i|$ stands for the size of the set B_i.

Notation: $\Im_P$ is the set of all SK-partitions.

The concepts of a generator set, an ideal and a filter are well known in lattice theory , Birkhoff [1].

Definition: G is said to be a generator set of a lattice L if every element of L is the upper
bound of elements of G.

Definition: Let ($L, \leq$) be a lattice. A subset A of L is called an <u>ideal</u>, if

1. $a, b \in A \Rightarrow a \vee b \in A$
2. $a \in A$ and $c \in L \ni c \leq a \Rightarrow c \in A$

Definition: Let ($L, \leq$) be a lattice. A subset H of L is called a <u>filter</u> if

1. $a, b \in H \Rightarrow a \wedge b \in H$
2. $a \in H$ and $c \in L \ni c \geq a \Rightarrow c \in H$.

Corresponding Author: Peter Chami, Department of Mathematics and Computer Science School, Faculty of Natural Sciences, The University of the West Indies, St. Augustine, Trinidad W.I., Tel: +868-789-7877, Fax: +868-645-7132,

Notation: For an SK-partition P of $F_q, \|P\|$ denotes the number of classes of P, and $[P]_i$ denotes the (i+1)th class, for $i = 0,1,...,$ $\|P\|-1$. Also if $P = \{B_0, B_1, ..., B_{m-1}\}$, then B_i^1 denotes $\left\{x \in B_i \middle| x < \frac{q}{2}\right\}$ and B_i^2 denotes $\left\{x \in B_i \middle| x > \frac{q}{2}\right\}$, i=1,2,…,m-1.

Notation: Z/qZ is the quotient group of integers modulo q with the discrete topology.

Definition: Given a partition P of F_q, we define a measure μ_P on Z/qZ as follows:
$\mu_P(i+qZ) = j$, if $i \in B_j, i = 0,1,...,q-1$.

THE CLASS-SIZE ORDERING

Definition: Let P and Q be elements of $\Im_P$ such that
$P = \{B_0, B_1, ..., B_{m-1}\}$ and
$Q = \{C_0, C_1, ..., C_{m'-1}\}; m, m' \in \{2,3,4,...\}$
where P is an SK-partition of F_q and Q is an SK-partition of $F_{q'}$;
$q, q' \in \{2,3,...\}$.
$P \leq_s Q \Leftrightarrow m \leq m'$ and the number of elements of F_q of weight ω with respect to $P \leq$ the number of elements of $F_{q'}$ of weights ω with respect to Q, $\omega = 0,1,...,m-1$.

Definition: Let μ_P be a measure on Z/qZ and μ_Q be a measure on $Z/q'Z$, where
$P = \{B_0, B_1, ..., B_{m-1}\}$ is an SK-partition of F_q, and $Q = \{C_0, C_1, ..., C_{m'} - 1\}$ is an SK-partition of $F_{q'}$. Also, let $M_P = \{\mu_P | P \in \Im\}$. We define an ordering on M_P as: For $\mu_P, \mu_Q \in M_P$,
$\mu_P \leq_\mu \mu_Q \Leftrightarrow$ number of elements of Z/qZ of measure $j \leq$ number of elements of $Z/q'Z$ of measure j
$j = 0,1,...,m-1$.

Note: Clearly $\mu_P \leq \mu_Q \Leftrightarrow P \leq_S Q$

Remark: Clearly, from the above definition $P \leq_S Q \Leftrightarrow |B_i| \leq |C_i|$, i = 0,1,…,m-1.

Theorem 3.1: $\leq_s$ is a partial ordering on $\Im_p$.

Proof: Let P, $Q \in \Im_p \ni P \leq_s Q$ and $Q \leq_s P$.
Then $m \leq m'$ and $m' \leq m$.
$\therefore m = m'$
(I) Also, $q \leq q'$ and $q' \leq q$
$q = q'$
(II) We also have
$|B_i| \leq |C_i|$ and $|C_i| \leq |B_i|; i = 0,1,...,m-1$
$\therefore |B_i| = |C_i|; i = 0,1,...,m-1$
(III) From (I), (II) and (III), it follows that P=Q. Hence $\leq_S$ is antisymmetric. Also, $\leq_S$ is reflexive and transitive. gsdf

Corollary 3.2: $\leq_\mu$ is a partial ordering on M_P.

The following example showed that $\leq_S$ is not linear.

Example. Let $P = \{B_0, B_1, B_2, B_3\}$
where $B_0 = \{0\}$
$B_1 = \{1,2,21,22\}$
$B_2 = \{3,4,5,6,17,18,19,20\}$
$B_3 = \{7,8,9,10,11,12,13,14,15,16\}$
and let $Q = \{C_0, C_1, C_2, C_3, C_4\}$
where $C_0 = \{0\}$
$C_1 = \{1,2,3,26,27,28\}$
$C_2 = \{4,5,6,23,24,25\}$
$C_3 = \{7,8,9,10,19,20,21,22\}$
$C_4 = \{11,12,13,14,15,16,17,18\}$
$|B_1| = 4 < 6 = |C_1|$
$\therefore Q \not\leq_S P$
also $|B_3| = 10 > 8 = |C_3|$
$\therefore P \not\leq_S Q$.

We devised a more convenient notation for an SK-partition which was expressed in terms of its class-sizes, and which would also uniquely determine the SK-partition.

Theorem 3.3: $a_1, a_2, a_3, ..., a_{m-1}$ are positive, even integers satisfying $a_1 \le a_2 \le .... \le a_{m-1}$ $\Leftrightarrow$ there exists a unique SK-partition $P = \{B_0, B_1, ..., B_{m-1}\}$ in $\Im_p \ni |B_i| = a_i; i = 1,2,...,m-1$.

Proof: Suppose that $a_1, a_2, ..., a_{m-1}$ are positive, even integers such that $a_1 \le a_2 \le .... \le a_{m-1}$

and that $q = 1 + \sum_{i=1}^{m-1} a_i$

Any class of an SK-partition of F_q is of the form $\{x, x+1, ..., x+a, q-(x+a), q-(x+a-1), ..., q-x\}$

Also, from condition (2) of an SK-partition, B_i and B_{i+1} can only be classes of an SK-partition $P = \{B_0, B_1, ..., B_{m-1}\}$ of F_q if the elements of B_i less than $\frac{q}{2}$ are all less than the elements of B_{i+1} less than $\frac{q}{2}, (i = 1,2,...,m-2)$.

The only partition $P = \{B_0, B_1, ..., B_{m-1}\}$ of F_q satisfying conditions (1) and (2) of an SK-partition must satisfy and following:

$$B_0 = \{0\}$$

$$B_1^{\ 1} = \left\{ x \in F_q \middle| 1 \le x \le \frac{a_1}{2} \right\}$$

$$B_i^{\ 1} = \left\{ x \in F_q \middle| 1 + \sum_{j=1}^{i-1} \frac{a_j}{2} \le x \le \sum_{j=1}^{i} \frac{a_j}{2} \right\}$$

$i = 2,3,...,m-1$.

P also satisfies condition (3) of an SK-partition since $|B_i| = a_i$ $(i = 1,2,...,m-1)$ and so $|B_i| \le |B_2| \le .... \le |B_{m-1}|$.

Hence P satisfies the three conditions of an SK-partition. Since $|B_i|$ is even for $i \in \{1,2,...,,m-1\}$ and $|B_1| \le |B_2| \le .... \le |B_{m-1}|$, $P \in \Im_p$.

Conversely, suppose that $P \in \Im_p \ni P = \{B_0, B_1, ..., B_{m-1}\}$ and $|B_i| = a_i (i = 1,2,...,m-1)$.

Since $P \in \Im_p$, then a_i's are positive and even for $i = 1,2,...,m-1$ and also $a_1 \le a_2 \le .... \le a_{m-1}$

Remark: From the previous theorem, any SK-partition in $\Im_p$ was determined by the sizes of its classes, so w the notation given below was used.

Notation: If $P \in \Im_p \ni P = \{B_0, B_1, ..., B_{m-1}\}$ and $|B_i| = a_i$ $(i = 1,2,...,m-1)$, we shall denote P by $((1, a_1, a_2, ..., a_{m-1}))$, when it is convenient.

THE LATTICES $(\Im_P, \le_s)$ and $(M_P, \le_\mu)$

We presented the l.u.b and g.l.b of any two elements in $(\Im_p, \le_s)$ and then established that it is a lattice.

Lemma 4.1: Let $P = \{B_0, B_1, ..., B_{m-1}\}$ and $Q = \{C_0, C_1, ..., C_{m'-1}\}, (m \le m')$ be any two elements of the poset $(\Im_p, \le_s)$.

The l.u.b of P and Q is G where $G = \{G_0, G_1, ..., G_{m'-1}\}$,

$|G_i| = \max\{|B_i|, |C_i|\}; i = 0,1,...,m-1$, and

$|G_i| = \max\{|B_{m-1}|, |C_i|\}; i = m, m+1, ..., m'-1$.

Also $G \in \Im_p$.

Proof: $|G_i| = \max\{|B_i|, |C_i|\} \le \max\{|B_{i+1}|, |C_{i+1}|\} = |G_{i+1}|$ for $i = 1,2,...,m-2$.

Also, for $i = m-1, m, m+1, m+2, ..., m'-2$

$|G_i| = \max\{|B_{m-1}|, |C_i|\} \le \max\{|B_{m-1}|, |C_{i+1}|\} = |G_{i+1}|$

From Theorem 3.3, G is an SK-partition.

G is an upper bound of P and Q since

$\|P\| = m \le m' = \|G\|$
$\|Q\| = m' = \|G\|$
$|B_i| \le |G_i|; i = 0,1,...,m-1.$
$|C_i| \le |G_i|; i = 0,1,...,m'-1.$

If K is another upper bound of P and Q, then $G \le_s K$, since $|G_i|$ is the smallest number satisfying
$|B_i| \le |G_i|; i = 0,1,...,m-1$
and
$|C_j| \le |G_j|; j = 0,1,...,m'-1.$
G is the l.u.b of P and Q.
$|G_i|$ is even for $(i = 1,2,...,m'-1)$ and $|G_i| \le |G_{i+1}|$ for $i = 1,2,...,m'-2$. $G \in \Im_p$.

Lemma 4.2: Let P and Q and $(\Im_p, \le_s)$ be as in Lemma 4.1. The g.l.b of P and Q is H, where
$H = \{H_0, H_1, ..., H_{m-1}\}$ and
$|H_i| = \min\{|B_i|, |C_i|\}, i = 0,1,...,m-1.$
Also $H \in \Im_p$.

Proof:
$|H_i| = \min\{|B_i|, |C_i|\} \le \min\{|B_{i+1}|, |C_{i+1}|\} = |H_{i+1}|$
$, i = 0,1,...,m-2.$
H is an SK-partition from Theorem 3.3. H is a lower bound of P and Q.
If L is another lower bound of P and Q, then $L \le_s H$, since $|H_i|$ is the largest number satisfying
$|H_i| \le |B_i|; i = 0,1,...,m-1$, and
$|H_j| \le |C_j|; j = 0,1,...m-1.$
Hence H is the g.l.b of P and Q. Clearly $|H_i|$ is even for $i = 1,2,...,m-1$ and $|G_i| \le |G_{i+1}|$ for $i = 1,2,...,m-2$. Hence $H \in \Im_p$.

From the foregoing the following result was obvious.

Theorem 4.3: $(\Im_p, \le_s)$ is a lattice.

Corollary 4.4: $(M_P, \le_\mu)$ is a lattice.

$(\Im_p, \le_s)$ is not distributive, as shown:

Example: Let P, Q and R be elements of $\Im_p$ such that
$P = ((1,2,2,6,6,8,8,10,14))$,
$Q = ((1,2,4,4,4,6,6,10))$ and $R = ((1,2,8,10))$.
Now, $Q \vee R = ((1,2,8,10,10,10,10,10))$
$P \wedge (Q \vee R) = ((1,2,2,6,6,8,8,10))$
$P \wedge Q = ((1,2,2,4,4,6,6,10))$
$P \wedge R = ((1,2,2,6))$
$(P \wedge Q) \vee (P \wedge R) = ((1,2,2,6,6,6,6,10))$
$\therefore P \wedge (Q \vee R) \ne (P \wedge Q) \vee (P \wedge R)$
$(\Im_p, \le_s)$ is not distributive.

Also, the following example showed that $(\Im_p, \le_s)$ is not modular.

Example: Let $P = ((1,2,4,4,6,8,10,12))$, $Q = ((1,6,8,10))$ and $R = ((1,2,2,4,6,6))$.
We have, $P \ge_s R$
$Q \vee R = ((1,6,8,10,10,10))$
$P \wedge (Q \vee R) = ((1,2,4,4,6,6,8))$
$P \wedge Q = ((1,2,4,4))$
$(P \wedge Q) \vee R = ((1,2,4,4,6,6))$
Hence $P \wedge (Q \vee R) \ne (P \wedge Q) \vee R$ and so $(\Im_P, \le_s)$ is not modular.

THE SUBLATTICES $(\Im_{P,m}, \le_s)$ and $(M_{P,m}, \le_\mu)$

Notation: For a fixed integer $m \in \{2,3,...\}$, let $\Im_{P,m} = \{P \in \Im_p | P \text{ has } m \text{ classes}\}$ and let $M_{P,m} = \{\mu_P \in M_P | P \text{ has } m \text{ classes}\}$
The following result was deduced.

Theorem 5.1: $(\Im_{P,m}, \le_s)$ is a sub lattice of $(\Im_P, \le_s)$ for each $m \in \{2,3,....\}$.

Remark: For any natural number n all codes over $(F_P)^n$ had the same maximum weight with respect to any element to any element of $\Im_{P,m}$. All codes have the same maximum distance with respect to any element of $\Im_{P,m}$.

Theorem 5.2: $(\Im_{P,m}, \leq_s)$ is a distributive lattice.

Proof: Let P, Q and R be fixed, arbitrary elements of $\Im_{P,m}$.

We show that $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$.

$\|P \wedge (Q \vee R)\| = \|(P \wedge Q) \vee (P \wedge R)\|$.

$|[P \wedge (Q \vee R)]_i| = |[(P \wedge Q) \vee (P \wedge R)]_i|$,

$i = 0,1,...,m-1$.

Let:

$$H_i = \left|\left[P \wedge (Q \vee R)\right]_i\right| = \min\left\{\begin{matrix}\left|[P]_i\right|, \\ \max\left\{\left|[Q]_i\right|, \left|[R]_i\right|\right\}\end{matrix}\right\}, i = 0,1,...,m-1.$$

$$K_i = \left|\left[(P \wedge Q) \vee (P \wedge R)\right]_i\right| = \max\left\{\begin{matrix}\min\left\{\left|[P]_i\right|, \left|[Q]_i\right|\right\}, \\ \min\left\{\left|[P]_i\right|, \left|[R]_i\right|\right\}\end{matrix}\right\}, i = 0,1,...,m-1.$$

$H_i = K_i; i = 0,1,2,...,m-1$.

If $|[P]_i| \geq |[Q]_i| \geq |[R]_i|$, then

$H_i = \min\{|[P]_i|, |[Q]_i|\} = |[Q]_i|$

$K_i = \max\{|[Q]_i|, |[R]_i|\} = |[Q]_i| = H_i$

For other relative sizes of $[P]_i, [Q]_i$ and $[R]_i$, similarly $H_i = K_i (i = 0,1,...,m-1)$.

Hence $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$.□

Corollary 5.3: $(M_{P,m}, \leq_\mu)$ is a distributive lattice

GENERATOR SETS OF ($\Im_{P,m}, \leq_s$) AND $(M_{P,m}, \leq \mu)$

Notation: Let $G_{P,m,1} = \left\{\begin{matrix}((1,2,2,...,2,\alpha_{m-1})) \\ \in \Im_{P,m} \left| \alpha_{m-1} = 2,4,6,... \right.\end{matrix}\right\}$,

$$G_{P,m,i} = \left\{\begin{matrix}((1,2,2,...,2,\alpha_{m-i},\alpha_{m-i+1},...,\alpha_{m-1})) \\ \in \Im_{P,m} \left| \alpha_{m-i} = \alpha_{m-i+1} = ... = \alpha_{m-1} = 4,6,8,... \right.\end{matrix}\right\}$$

for $i \in \{2,3,...,m-1\}$

and $G_{P,m} = \bigcup_{i=1}^{m-1} G_{P,m,i}$

Lemma6.1 $G_{P,m}$ **is a generator set of** $(\Im_{P,m}, \leq_s)$ where $(m = 2,3,...)$

Proof: We showed that any element of $\Im_{P,m}$ is the upper bound of elements of $G_{P,m}$.

Let $((1,a_1,a_2,...,a_{m-1}))\begin{pmatrix}a_i \in \{2,4,6,...\}; \\ i = 1,2,...,m-1\end{pmatrix}$ be fixed, arbitrary elements of $\Im_{P,m}$.

each SK-partition on the R.H.S is in $G_{P,m}$.

In view of the above, we infered the following result:

Theorem 6.2: G_P is a generator set for $(\Im_P, \leq_s)$, where $G_P = \bigcup_{m=2}^{\infty} G_{P,m}$.

Notation: Let $G_{\mu,P,m} = \{\mu_P \in M_P | P \in G_{P,m}\}$ and $G_{\mu,P} = \{\mu_P | P \in G_P\}$

Corollary 6.3: $G_{\mu,P}$ is a generator set for $(M_P, \leq_\mu)$.

Notation: Let K be infinite, increasing sequence of positive, even numbers, namely, $k_1, k_2, ...$

Also let

$$H_{P,m,k,i} = \left\{\begin{matrix}((1,2,2,...,2,a_{m-i},a_{m-i+1},...,a_{m-1}))\Im_{P,m} \\ \text{where } a_{m-i} = a_{m-i+1} = ... = a_{m-1}\end{matrix}\right\}$$ for $i \in \{1,2,...,m-2\}$,

$$H_{P,m,K,m-1} = \left\{\begin{matrix}((1,a_1,a_2,...,a_{m-1})) \\ \in \Im_{P,m} \left| \begin{matrix}a_1 = a_2 \\ = ... = a_{m-1} = 2,4,6,...\end{matrix} \right.\end{matrix}\right\}$$

and

$$H_{P,m,K} = \bigcup_{i=1}^{m-1} H_{P,m,K,i}$$

Theorem 6.4: $G_{P,m}, H_{P,m,K_1}, H_{P,m,K_2}, ...$ form an infinite chain of generators for $(\Im_{P,m}, \leq_s)$, where K_h is an infinite, increasing sequence of positive, even numbers and K_{h+1} is an infinite subsequence of $K_h, (h = 1,2,...)$.

Proof: Let j be a fixed, arbitrary element of $\{1,2,...\}$. We showed that H_{P,m,K_j} is a generator of $(\Im_{P,m}, \leq_s)$ by proving that any element of $G_{P,m}$ was obtained by performing lattice operations on elements of H_{P,m,K_j}.

Let $K_j = k_{j1}, k_{j2},...$ be an increasing sequence of positive, even numbers and let $A = ((1,2,2,...,2,a,a,...,a))$ be an arbitrary element of $G_{P,m}$ for some fixed, positive integer such that $A \notin G_{P,m,m-1}$. Then $A \in G_{P,m,i}$ for some $i \in \{1,2,...,m-2\}$.

$\therefore A = ((1,2,2,...,2,a,a,...,a))$

←------→←-------→

$(m-1-i)$ *times* (i *times*)

$= ((1,2,2,...,2,b,b,...,b)) \wedge ((1,a,a,...,a))$

←(m-1-i) twos → ← i b's → ←(m-1) a's →

where $b \in \{k_{j1}, k_{j2},...\} \ni b \geq a$.

$((1,2,2,.....,2,b,b,...b)) \in H_{P,m,k_j,i}$

←(m-1-i) twos→ ←i b's→

and $((1,a,a,.......,a)) \in H_{P,m,K_j,m-1}$

←(m-1) times→

We have shown that A is the g.l.b of elements of H_{P,m,K_j}.

If $B \in G_{P,m,m-1}$, then $B \in H_{P,m,K_j}$.

Any element of $G_{P,m}$ either already was an element of H_{P,m,K_j} or could be obtained by taking the g.l.b. of two elements of H_{P,m,K_j}.

Notation: Let $H_{\mu,P,m,k} = \{\mu_P \in M_P | P \in H_{P,m,k}\}$.

Corollary 6.5: $G_{\mu,P,m}, H_{\mu,P,m,k_1}, H_{\mu,P,m,k_2},...$ form an infinite chain of generators for $(M_{P,m}, \leq_\mu)$.

IDEALS AND FILTERS OF $(\Im_p, \leq_s)$ and $(M_P, \leq_\mu)$

We next considered SK-partitions with bounded class sizes. In the next theorem, we showed that some sets of such zpartitions are ideals of $(\Im \ , \leq_s)$.

Theorem 7.1: Every ideal H of $(\Im_P, \leq_s)$ has one of the forms:

$(i)\, H = \Im_P$.

$$(ii)\, H = \left\{ P \in \Im_P \left| \begin{array}{l} P = \{B_0, B_1, ..., B_{m-1}\}, m \leq d, \\ \text{and either } (1) \\ |B_i| \leq b_i\,(i = 1,2,...,h), \text{if } h \leq m-1 \\ \\ \text{or } (2) |B_i| \leq b_i\,(i = 1,2,...,m-1), \text{if } h > m-1 \end{array} \right. \right\}$$

where h and d are fixed, arbitrary, positive integers such that $h \leq d-1$ and $b_1, b_2, ..., b_h$ are fixed, arbitrary, positive, even integers satisfying $b_1 \leq b_2 \leq .... \leq b_h$.

(iii) H has the same form as in (ii), except m is not bounded.

Proof: Let H be an ideal of $(\Im_P, \leq_s), d = \max\{m | P = \{B_0, B_1, ..., B_{m-1}\} \in H\}$

$$\text{and } h = \max\left\{ r \left| \begin{array}{l} P = \{B_0, B_1, ..., B_{m-1}\} \in H, \text{ and either} \\ (1) |B_1|, |B_2|, ..., |B_r| \\ \text{are bounded above if } r \leq m-1 \\ \text{or} \\ (2) |B_1|, |B_2|, ..., |B_{m-1}| \\ \text{are bounded above if } r > m-1 \end{array} \right. \right\}$$

H takes different forms, depending on whether h exists and d exists.

(i) If h does not exist, then d does not exist and $H = \Im_P$.

(ii) (a) If h and d exist, and $h < d-1$, let

$$b_i = \max\left\{ \begin{array}{l} |B_i| \, P = \{B_0, B_1, ..., B_{m-1}\} \\ \in H, i = 1,2,...,h \end{array} \right\}$$

From the definition of $b_i\,(i = 1,2,....,h), \exists\, Q_i \in H \ni Q_i = \{Q_{i0}, Q_{i1}, Q_{i2}, ..., Q_{i(m_i-1)}\}$, $m_i - 1 \geq i$ and $|Q_{ii}| = b_i\,(i = 1,2,...,h)$.

Let $D = Q_1 \vee Q_2 \vee ... \vee Q_h$

From property (1) of an ideal, $D \in H$. Clearly $D = \{D_0, D_1, D_2, ..., D_{n_1-1}\}$, for some integer $n_1 \ni h \le n_1 - 1$, and $|D_i| = b_i \; (i = 1,2,3,...,h)$.

$\therefore b_1 \le b_2 \le .... \le b_n$.

Let

$$K = \left\{ P \in \Im_P \left| \begin{array}{l} P = \{B_0, B_1, ..., B_{m-1}\}, m \le d, \text{ and either} \\ \quad (1)|B_i| \le b_i \; (i = 1,2,....,h), \text{ if } h \le m-1, \\ \text{or } (2)|B_i| \le b_i (i = 1,2,...,m-1 \text{ if } h > m-1 \end{array} \right. \right\},$$

We proved that $H = K$.

Let $L = \{L_0, L_1, ..., L_{n_2-1}\}$ be an arbitrary element of H, for some positive integer n_2.

Then either $(1)|L_i| \le b_i \; (i = 1,2,...,h)$ if $h \le n_2 - 1$

or $(2)|L_i| \le b_i (i = 1,2,..., n_2 - 1)$, if $h > n_2 - 1$

from the definition of $b_i \; (i = 1,2,...,h)$.

$\therefore L \in H \Rightarrow L \in K$,

$\therefore H \subseteq K \qquad\qquad (I)$

Let M be an arbitrary element of $K \ni \|M\| \le h$.

Then $M \le_s D$.

Since $D \in H$, from property (ii) of an ideal $M \in H$.

Now, let $N = \{N_0, N_1, ..., N_{n_3-1}\}$ (where n_3 is a positive integer) be an arbitrary element of $K \ni \|N\| > h$.

Since only the first $(h+1)$ classes of elements of H are bounded (for those elements of H having $h+1$ or more classes),

$\exists\, T = \{T_0, T_1, ..., T_{n_4-1}\}$

$\in H, (n_3 \le n_4), \ni |N_{n_3-1}| \le |T_{h+1}|$.

Also, $\exists\, U = \{U_0, U_1, ..., U_{n_5-1}\} \in H \ni n_3 \le n_5$, from the definition of d.

Let $V = D \vee T \vee U$.

From property (i) of an ideal, $V \in H$.

Since $N \le_s V$, from property (ii) of an ideal, $N \in H$.

$\therefore N \in K \Rightarrow N \in H$

$\therefore K \subseteq H$.

(II)

From (I) and (II)

$$H = K$$

(b) If $H = d - 1$, the proof is similar .

If h exists and d does not exist, the proof is also similar to the proof in (ii).

Corollary 7.2: If $H_{\mu_P} = \{\mu_P \in M_P | P \in H\}$ where H is an ideal of $(\Im_P, \le_s)$, then $H_{\mu_{PP}}$ is an ideal of $(M_P, \le_\mu)$.

The next result was expressed in terms of filters of $(\Im_P, \le_S)$.

Theorem 7.3: Every filter H of $\{(\Im_P, \le_s)\}$ has the form: $H = \{P \in \Im_P \,|\, P \ge D\}$ for some fixed element D of $\Im_P$.

Proof: Let H be a filter of $(\Im_p, \le_s)$, and let $h + 1 = \min\{m \in N | P = \{B_0, B_1, ..., B_{m-1}\} \in H\}$.

Also let $b_i = \min\{|B_i| \in N | P = \{B_0, B_1, ..., B_{m-1}\} \in H\}$, $i = 1, 2, ..., h.$

From the definition of $h+1$, $\exists$ some element $C = \{C_0, C_1, ..., C_h\} \in H$, and from the definition of $b_i \; (i = 1,2,....,h), \exists Q_i \in H \ni$

$Q_i = \{Q_{i0}, Q_{i1}, Q_{i2}, ..., Q_{i(m_i-1)}\}$,

$|Q_{ii}| = b_i (i = 1,2,..., h)$, and $h \le m_i - 1$.

Let $D = C \wedge Q_1 \wedge Q_2 \wedge ... \wedge Q_h$.

From property (i) of a filter $D \in H$.

$D = \{D_0, D_1, D_2, ..., D_h\}$,

where $|D_i| = b_i \; (i = 1,2,...,h)$ and $H = \{P \in \Im_P \,|P \ge D\}$

Corollary 7.4: Every filter K of $(M_P, \le_\mu)$ has the form $K = \{\mu_P \in M_P | P \ge D\}$, where D is a fixed element of $\Im_P$.

REFERENCES

1. Birkhoff G, 1948, Lattice Theory, American Mathematical Society; Rhode Island.

2. Kaushik ML, 1978, Burst-error-correcting codes with weight constraints under a new metric. J. Cybernetics, 8, 183-202.
3. Kaushik ML, 1979a, Single error and burst error correcting codes through a new metric. J. Cybernetics, 8, 1-15.
4. Kaushik ML, 1979b, Necessary and sufficient number of parity-checks correcting random errors and burst with weight constraints under a new metric. J. Cybernetics, 9, 81-90
5. Kaushik ML, 1979c, Random error detecting and burst error correcting codes under a new metric. Indian J. Pure and Appl. Math., 10, 1460-1468.
6. Kaushik ML, 1979d, A new metric in the study of error correcting codes. Ph. D. thesis, University of Delhi.
7. Kaushik ML, 1981, Channels and bounds for class-metric codes. Rev. Roum. Math. Pures et Appl., 26, 1345-1350.
8. Maharan D, 1982, Orthogonality measures: an example. Annals of Probability, Vol. 10, No. 3, 879-880.
9. Niederreiter H and Sookoo N, 2000, Partial densities on the group of integers. Arch. Math. (Brno) Tomus 36, 17-24.
10. Schmidt KD, 1986, Decompositions of vector measures in Riesz spaces and Banach lattices, Proc. Edinburgh Math. Soc. (2) 29, 23-39.
11. Sharma BD and Dial G, 1987, Some tighter bounds on code size with Sharma-Kaushik metrics. Presented at the Intern. Conf. on Math., Mao, Menorca, June 15-17.
12. Sharma BD and Kaushik ML, 1977, Error correcting codes through a new metric. 41st Annual Conf. of Inter. Stat. Inst., New Delhi.
13. Sharma BD and Kaushik ML, 1979, Limited intensity random and burst error codes with class-weight considerations. Electronische Infor. und Kyber., 15, 315-321.
14. Sharma BD and Kaushik ML, 1986, Algebra of Sharma and Kaushik's metric inducing partitions of Z_q. J. Comb. Infor. And Sys. Sci., 11, 19-32.