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Weyl's Type Theorems for Quasi-Class A Operators

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Abstract: A variant of Weyl theorem for a class of quasi-class A acting on an infinite complex Hilbert space were discussed. If the adjoint of T is a quasi-class A operator, then the generalized a-Weyl holds for f(T) , for every function that analytic on the spectrum of T. The generalized Weyl theorem holds for a quasi-class A was proved. Also, a characterization of the Hilbert space as a direct sum of range and kernel of a quasi-class A was given. Among other things, if the operator is a quasi-class A, then the B-Weyl spectrum satisfies the spectral theorem was characterized.

Key words: Single valued Extension property, Fredholm theory, Browder's spectrum theory

INTRODUCTION

Throughout this study let B(H) and K(H), denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space H. If $T\in B(H)$ we shall write ker(T) and ran(T) for the null space and range of T, respectively. Also, let $\alpha(T)$: = dim ker(T), $\beta(T)$: = co dim ran (T) and let $\sigma(T),\sigma_a(T)\sigma_p(T)$ denote the spectrum, approximate point spectrum and point spectrum of T, respectively. An operator $T \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space and its range has finite co dimension. The index of a Fredholm operator is given by

$$i(T) := \alpha(T) - \beta(T)$$

T is called Weyl if it is Fredholm of index 0 and Browder if it is Fredholm of finite ascent and descent.

The essential spectrum σ_e (T), the Weyl spectrum σ_W (T) and the Browder spectrum σ_b (T) of T are defined by

$$\sigma_e(T) = \{\lambda\in C : T-\lambda I \text{ is not Fredholm}\},$$

$$\sigma_W(T) = \{\lambda\in C : T-\lambda I \text{ is not Weyl}\},$$

and

$$\sigma_b(T) = \{\lambda\in C : T-\lambda I \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e(T) \subseteq \sigma_W(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \cup \operatorname{acc}\sigma(T),$$

Where, we write accK for the accumulation points of $K\subseteq C$. If we write isoK = K - accK then we let

$$E_0(T) := \{\lambda\in \operatorname{iso}\sigma(T) : 0 < \alpha(T-\lambda I) < \infty\}$$

for the isolated eigenvalues of finite multiplicity and

$$p_0(T) := \sigma(T) - \sigma_b(T) \qquad (1.1)$$

for the Riesz points of T. Then (1.1) with the help of "Punctured neighborhoods Theorem"

$$\operatorname{iso}\sigma(T) - \sigma_e(T) = \operatorname{iso}\sigma(T) - \sigma_W(T) = p_0(T) \subseteq E_0(T).$$

Definition 1: [6] We say that Weyl's theorem holds for $T\in B(H)$ if

$$\sigma(T) - \sigma_W(T) = E_0(T),$$

and we shall say that Browder's theorem holds for $T\in B(H)$ if

$$\sigma(T) - \sigma_W(T) = p_0(T).$$

Evidently Weyl's theorem implies Browder's theorem. Let us denote by:

$$\Phi_+(H) = \{T\in B(H) : \alpha(T) < \infty \text{ and ran}(T) \text{ is closed}\}$$

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the class of all upper semi-Fredholm operators and

$$\Phi_{-}(H) = \{T \in B(H) : \beta(T) < \infty\}$$

the class of all lower semi-Fredholm operators. The class of all semi-Fredholm operators is defined by $\Phi_{\pm}(H) = \Phi_{+}(H) \cup \Phi_{-}(H)$, whilst the class of all Fredholm operators is defined by $\Phi(H) = \Phi_{+}(H) \cap \Phi_{-}(H)$. The ascent a: = a(T) of an operator T is the smallest non-negative integer s such that $\ker(T^s) = \ker(T^{s+1})$. If such integer does not exist we put $a(T) = \infty$.

Analogously, the descent d = d (T) of an operator T is the smallest non-negative integer t such that $\mathrm{ran}(T^t) = \mathrm{ran}(T^{t+1})$ and if such integer does not exist we put $d(T) = \infty$. It is well-known that if a(T) and d(T) are both finite then a(T) = d(T) [7,proposition 1.49]. Two other important classes of operators in Fredholm theory are the class of all upper semi-Browder operators

$$B_{+}(H) := \{T \in \Phi_{+}(H) : a(T) < \infty\}$$

and the class of all lower semi-Browder operators

$$B_{-}(H) := \{T \in \Phi_{-}(H) : d(T) < \infty\}.$$

The class of all Browder operators is defined by $\mathrm{Bro}(H) := B_{+}(H) \cap B_{-}(H)$. Note that if $T \in B_{+}(H)$ then the index is defined by $i(T) = \alpha(T) - \beta(T)$ is less than or equal to 0, whilst if $T \in B_{-}(H)$, then $i(T) \geq 0$,[14]. The class of all Weyl Operators W(H) is defined by

$$W(H) = \{T \in \Phi(H) : i(T) = 0\}.$$

Note that $\mathrm{Bro}(H) \subseteq W(H)$, since every Fredholm operator with finite ascent and finite descent has necessary index 0,[1,9,10]. The classes of operators defined above motivate the definition of several spectra. The essential approximate point spectrum is

$$\sigma_{ea}(T) := \bigcap\{\sigma_a(T+K) : K \in K(H)\},$$

and

$$\sigma_{ba}(T) := \bigcap\{\sigma_a(T+K) : TK = KT, K \in K(H)\},$$

is the Browder essential approximate point spectrum. It is well-known that $\sigma_{ea}(T) = \{\lambda \in C : T - \lambda I \notin B_{+}(H)\}$.

Definition 2:[1] We say that a-Browder's holds for T if

$$\sigma_{ea}(T) = \sigma_{ba}(T).$$

It is known that if $T \in B(H)$ then a-Browder's theorem implies Browder's theorem. In [8], the authors proved that Weyl's theorem holds for quasi-classA, in this paper, we prove that generalized Weyl's holds for quasi-class A operators.

RESULTS

Definition 3: An operator $T \in B(H)$ is said to be quasi-class A if

$$T^*|T^2|T \geq T^*|T|^2 T.$$

The class of quasi-class A introduced and studied by Jeon and Kim [15], for more interesting properties the reader should refer to [8,15].

Lemma 4: Let $T \in B(H)$ be a quasi-class A. Then $H = \mathrm{ran}(T) \oplus \ker(T)$. Moreover$T_1$, the restriction of T to ran(T) is one-one and onto.

Proof: Suppose that

$$y \in \mathrm{ran}(T) \cap \ker(T) \text{ then } y = Tx$$
$$\text{for some } x \in H \text{ and } Ty = 0.$$

It follows that $T^2x = 0$ However, a(T) = 1 and so $x \in \ker(T^2) = \ker(T)$. Hence $y = Tx = 0$ and so $\mathrm{ran}(T) \cap \ker(T) = \{0\}$. Also, $T(\mathrm{ran}(T)) = \mathrm{ran}(T)$. If $x \in H$ there is $u \in \mathrm{ran}(T)$ such that

Tu = Tx. Now if z = x-u then $T_z = 0$ Hence $H = \mathrm{ran}(T) \oplus \ker(T)$. Since d(T) = 1, T maps ran(T) onto itself. If $y \in \mathrm{ran}(T)$ and Ty = 0 then $y \in \mathrm{ran}(T) \cap \ker(T) = \{0\}$. Hence T_1 is one-one and onto.

Recall that an operator $S \in B(H)$ is said to be quasiaffine transform of T (abbreviate $S \prec T$) if there is a quasiaffinity X such that XS = TX.

Definition 5:[13] Let $\mathrm{Hol}(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. We say that $T \in B(H)$ has the single-valued extension property (SVEP) if for every open set $U \subseteq \mathbb{C}$ the only analytic function $f : U \to H$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ is the constant function $f = 0$.

It is well-known that $T \in B(H)$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} - \sigma(T)$. Moreover, from the identity theorem for analytic function it easily follows that $T \in B(H)$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$ [16]. In [18,proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Lemma 6: If $T \in B(H)$ is a quasi-class A operator and $S \prec T$. Then S has SVEP.

Proof: Since T is a quasi-class A operator and it has a SVEP, then the result follows from [6].

For $T \in B(H)$, it is known that the inclusion $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ holds for every $f \in Hol(\sigma(T))$, with no restriction on T. The next theorem shows that for quasi-class A operators the spectral mapping theorem holds for the essential approximate point spectrum. □

Theorem 7: If $T \in B(H)$ is a quasi-class A operator. Then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ holds for every $f \in Hol(\sigma(T))$.

Proof. Let $f \in Hol(\sigma(T))$. It suffices to show that $\sigma_{ea}(f(T)) \supseteq f(\sigma_{ea}(T))$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$ then $f(T) - \lambda I \in \Phi_+(H)$ and $i(f(T) - \lambda I) \le 0$ and $f(T) - \lambda I = c(T - \alpha_1 I)\ldots(T - \alpha_n)g(T)$, Where, $c, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and g(T) is invertible. If T is a quasi-class A, then $\sum_{j=1}^{n} i(T - \alpha_j I) \le 0$ and $i(T - \alpha_j I) \le 0$ for each $j = 1, \ldots, n$. Therefore $\lambda \notin f(\sigma_{ea}(T))$. This completes the proof. □

Definition 8:[12] For $T \in B(H)$ and closed subset F of $\mathbb{C}$ the glocal spectral is

$$\aleph_T(F) = \{x \in H : \exists \text{ analytic function } f : \mathbb{C} - F \to H \text{ such that } (\lambda I - T)f(\lambda) = x,\ \forall x \in \mathbb{C} - F\}.$$

Definition 9:[12] The quasinilpotent part $H_0(T - \lambda I)$ and the analytic core $K(T - \lambda I)$ are defined by $H_0(T - \lambda I) := \{x \in H : \lim_{n\to\infty} \|(T - \lambda I)^n x\|^{\frac{1}{n}} = 0\}$, and

$$K(T - \lambda I) = \{x \in H : \text{there exists a sequence } \{x_n\} \subset H \text{ and } \delta > 0 \text{ for which } x = x_0,\ (T - \lambda I)x_{n+1} = x_n \text{ and } \|x_n\| \le \delta^n \|x\| \text{ for all } n = 1, 2, \ldots\},$$

respectively.

Note that $H_0(T - \lambda I)$ and $K(T - \lambda I)$ are generally non-closed hyper-invariant subspaces of $T - \lambda I$ such that $(T - \lambda I)^{-p}(0) \subseteq H_0(T - \lambda I)$ for all $p = 0, 1, 2, \ldots$ and $(T - \lambda I)K(T - \lambda I) = K(T - \lambda I)$. For more information about this subject the reader should refer to [11,12].

Recall that generalized Weyl's theorem (g-Weyl's) holds for T if $\sigma(T) - \sigma_{BW}(T) = E(T)$, Where, E(T) denotes the isolated points λ of $\sigma(T)$, which are eigenvalues (no restriction on multiplicity) and $\sigma_{BW}(T)$ is the set of all complex numbers $\lambda$for which $T - \lambda I$ is not B-Weyl's. Berkani [3,proposition 3.2] has called an operator $T \in B(H)$ is B-Fredholm if there exists a natural number n for which the induced operator $T_n : ran(T^n) \to ran(T^n)$ is Fredholm in the usual sense and B-Weyl's" if in addition T_n has zero index. Berkani [3,corollary 3.3] has shown that, if g-Weyl's theorem holds for T then so does Weyl's theorem.

For the sake of simplicity of notation we introduce the abbreviations gaW, aW, gW and W to signify that an operator $T \in B(H)$ (which is usually understood) obeys generalized a-Weyl's theorem, a- Weyl's theorem, generalized Weyl's theorem and Weyl's theorem, respectively.

Analogous meaning is attached to the abbreviations gaB, aB, gB and B with respect to Browder's theorem. In the following diagrams, arrows signify implications between various Weyl's and Browder's theorems [2,4,5,20].

$$\begin{array}{ccccccc} gaB & \leftarrow & gaW & \rightarrow & gW & \rightarrow & gB \\ \updownarrow & & \downarrow & & \downarrow & & \updownarrow \\ aB & \leftarrow & aW & \rightarrow & W & \rightarrow & B \end{array}$$

Theorem 10: If $T \in B(H)$ is a quasi-classA operator. Then $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ holds for every $f \in Hol(\sigma(T))$.

Proof: It is suffices to show $\sigma_{BW}(f(T)) \supseteq f(\sigma_{BW}(T))$ since the other inclusion holds for every $f \in Hol(\sigma(T))$ with no restriction on T. Let $\mu \in \sigma_{BW}(T)$, and $f \in Hol(\sigma(T))$. Since $\sigma(T)$ is a compact subset of $\mathbb{C}$, the function $f(z) - f(\mu)$ possesses at most a finite number of zeros in $\sigma(T)$. So

$$f(T) - f(\mu I) = (T - \mu I)^m \prod_{j=1}^{n} (T - \lambda_j I) g(T),$$

Where, $\mu, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and g(T) is an invertible operator. So $g(T)^{-1}$ is a B-Weyl's operator. If $f(T) - f(\mu I)$ is B-Weyl's operator, by [3] applied to $f(T) - f(\mu I)$ and $g(T)^{-1}$ we have

$$(f(T)-f(\mu I))g(T)^{-1}=(T-\mu I)^m\prod_{j=1}^{n}(T-\lambda_j I)$$

is a B-Weyl's operator. So from [3] $T-\mu I$ is B-Weyl's, a fact which contradicts our assumption. Hence $f(\mu)\in\sigma_{BW}(f(T))$ and $f(\sigma_{BW}(T))\subseteq\sigma_{BW}(f(T))$.

Theorem 11: Let T be a quasi-class A operator. Then generalized Weyl's theorem holds for f(T) for all $f\in Hol(\sigma(T))$.

Proof: Since T is isoloid in $\sigma(T)$ by[8,lemma 1.8] and has SVEP, then it suffices to prove that generalized Weyl's theorem holds for T. We shall show that $\sigma(T)-\sigma_{BW}(T)=E(T)$. Let $\lambda\in\sigma(T)-\sigma_{BW}(T)$, then $T-\lambda I$ is B-Weyl's. Then by [3,theorem 2.7] there exists two closed subspaces N and M of H such that $H=M\oplus N$, $T_1=(T-\lambda I)|_M$ is Weyl's operator, $T_2=(T-\lambda I)|_N$ is nilpotent and $T-\lambda I=T_1\oplus T_2$. We have two possibilities: either $\lambda\in\sigma(T|_M)$ or $\lambda\notin\sigma(T|_M)$.

Case I: $\lambda\in\sigma(T|_M)$. Since $T|_M$ is quasi-class A , then Weyl's theorem holds for $T|_M$ and so if $\lambda\in\sigma(T|_M)$, then $\lambda\in E_0(T|_M)\subset iso\sigma(T|_M)$. Since $T-\lambda I=(T|_M-\lambda I|_M)\oplus T_2$ and T_2 is nilpotent , $\sigma(T_1)-\{0\}=\sigma(T-\lambda I)-\{0\}$ and $\lambda\in iso\sigma(T)$. This implies that $\lambda\in E_0(T)\subset E(T)$.

Case II: $\lambda\notin\sigma(T|_M)$. Then λ is a pole of T which implies that $\lambda\in E(T)$. Conversely, let $\lambda\in E(T)$. Let P be the spectral projection associated with λ, then $ran(P)=H_0(T-\lambda I), ker(P)=K(T-\lambda I)$,

$$H_0(T-\lambda I)\neq 0, H=H_0(T-\lambda I)\oplus K(T-\lambda I),$$

$K(T-\lambda I)$ is closed subspace[16,19], Since $0\neq ker(T-\lambda I)\subset H_0(T-\lambda I)$, λ is a pole of the resolvent $\Re_\lambda(T)=(T-\lambda I)^{-1}$, then by [16] there is some q>0 such that the space $(T-\lambda I)^{-q}(0)$ is non-zero and complemented by a closed T-invariant subspace $ran((T-\lambda I)^q)\subset ran(T-\lambda I)$. Hence $T-\lambda I$ is B-Weyl's, i.e., $\lambda\notin\sigma_{BW}(T)$.

A bounded linear operator T is called a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T . Note that every a-isoloid operator is isoloid and the converse is not true in general.

Theorem 2.4 of [21]affirms that if T^* or T has the SVEP and if T is a-isoloid and generalized a-Weyl's holds for T then generalized a-Weyl's theorem holds for f(T), for every $f\in Hol(\sigma(T))$. If T^* is quasi-class A, then we have:

Theorem 12: Let T^* be a quasi-class A operator. Then generalized a-Weyl’s theorem hold for T.

Proof: Since T^* has SVEP then $\sigma(T)=\sigma_a(T)$ and consequently $E(T)=E^a(T)$. Let $\lambda\notin\sigma_{SBF_+^-}(T)$ be given, then $T-\lambda I$ is semi-B-Fredholm and $i(T-\lambda I)\leq 0$. Then [17,proposition 1.2] implies that $i(T-\lambda I)=0$ and consequently $T-\lambda I$ is B-Weyl's. Hence $\lambda\notin\sigma_{BW}(T)$. So it follows from [21,theorem 3.1] that $\lambda\in E(T)=E^a(T)$. For the converse, let $\lambda\in E^a(T)$. Then $\lambda\in iso\sigma_a(T)$. Since T^* has the SVEP, we have $\sigma(T)=\sigma_a(T)$. Hence $\bar{\lambda}\in\sigma(T^*)$. Now we represent T^* as the direct sum $T^*=T_1\oplus T_2$, Where, $\sigma(T_1)=\{\bar{\lambda}\}$ and $\sigma(T_2)=\sigma(T_1)-\{\bar{\lambda}\}$. Since T is quasi-class A then so does T_1 and so we have two cases:

Case I: ($\bar{\lambda}=0$): Then T_1 is quasinilpotent. Hence it follows that T_1 is nilpotent. Since T_2 is invertible, then T^* is a B-Weyl's.

Case II: ($\bar{\lambda}\neq 0$): Since $\sigma(T_1)=\{\bar{\lambda}\}$, then $T_1-\bar{\lambda}I$ is nilpotent and $T_2-\bar{\lambda}I$ is invertible. Hence it follows from [21,theorem 3.1] that $T^*-\bar{\lambda}I$ is B-Weyl's. Thus in any case $\lambda\in\sigma_a(T)-\sigma_{SBF_+^-}(T)$.

Theorem 13: Let $T\in B(H)$ and T or T* is a quasi-class A. Then the generalized a-Browder’s theorem holds for T.

Proof: The proof is a consequence immediate of[8,2].

CONCLUSION

It can be shown that if T* is a quasi-class A then the generalized a-Browder’s theorem holds for f(T) for every $f\in Hol(\sigma(T))$.

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