Journal of Mathematics and Statistics 9 (4): 310-324, 2013
ISSN: 1549-3644

doi:10.3844/jmssp.2013.310.324 Published Online 9 (4) 2013 (http://www.thescipub.com/jmss.toc)

THE THEORY OF COMPLEX PROBABILITY AND THE FIRST ORDER RELIABILITY METHOD

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Received 2013-09-08; Revised 2013-10-06; Accepted 2013-10-09

ABSTRACT

The Kolmogorov's system of axioms can be extended to encompass the imaginary set of numbers and this by adding to the original five axioms an additional three axioms. Hence, any experiment can thus be executed in what is now the complex set C (Real set R with real probability + Imaginary set M with imaginary probability). The objective here is to evaluate the complex probabilities by considering supplementary new imaginary dimensions to the event occurring in the "real" laboratory. Whatever the probability distribution of the input random variable in R is, the corresponding probability in the whole set C is always one, so the outcome of the random experiment in C can be predicted totally. The result indicates that chance and luck in R is replaced now by total determinism in C. This new complex probability model will be applied to the concepts of degradation and the Remaining Useful Lifetime (RUL), thus to the field of prognostic based on reliability. Therefore, an example of Young modulus will be applied and the First Order Reliability Method (FORM) analysis will be used for this purpose.

Keywords: Complex Probability, Prognostic, Degradation, Remaining Useful Lifetime, Young Modulus, First Order Reliability Method (FORM), Failure Probability

1. INTRODUCTION

Abou Jaoude *et al*. (2010); Abou Jaoude (2013a; 2013b; 2005; 2007; 2012); Bell (1992); Benton (1996); Boursin (1986); Chan Man Fong *et al*. (1997); Cheney and Kincaid (2004); Dacunha-Castelle (1996); Dalmedico Dahan *et al*. (1992); Dalmedico Dahan and Peiffer (1986); Ekeland (1991); Feller (1968); Finney *et al*. (2004); Gentle (2003); Gerald and Wheatley (1999); Gleick (1997) and Greene (2000) firstly, the Extended Kolmogorov's Axioms (EKA for short) paradigm can be illustrated by the following figure (**Fig. 1**).

In engineering systems, the remaining useful lifetime prediction is related deeply to many factors that generally have a chaotic behavior which decreases the degree of our knowledge of the system.

As the Degree of Our Knowledge (DOK for short) in the real universe R is unfortunately incomplete, the extension to the complex universe C includes the contributions of both the real universe R and the imaginary universe M. Consequently, this will result in a complete and perfect degree of knowledge in C = R+M (Pc = 1). In fact, in order to have a certain prediction of any event it is necessary to work in the complex universe C in which the chaotic factor is quantified and subtracted from the Degree of Our Knowledge to lead to a probability in C equal to one (Pc^2 = DOK-Chf = 1). Thus, the study in the complex universe results in replacing the phenomena that used to be random in R by deterministic and totally predictable ones in C.

This hypothesis is verified in a previous study and paper by the mean of many examples encompassing both discrete and continuous distributions.

From the Extended Kolmogorov's Axioms (EKA), we can deduce that if we add to an event probability in the real set R the imaginary part M (like the lifetime variables) then we can predict the exact probability of the remaining lifetime with certainty in C (Pc = 1).

We can apply this idea to prognostic analysis through the degradation evolution of a system. As a matter of fact, prognostic analysis consists in the prediction of the remaining useful lifetime of a system at any instant t_0 and during the system functioning.

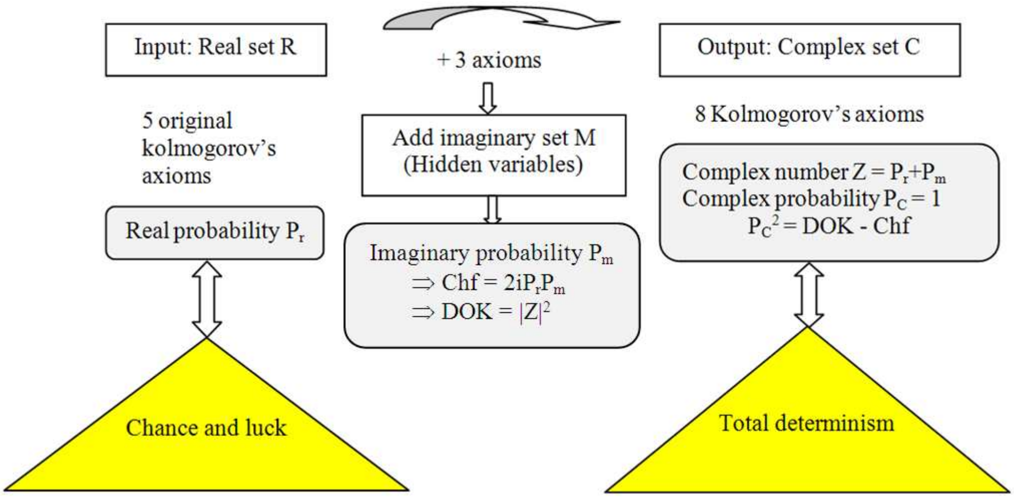

Fig. 1. EKA paradigm

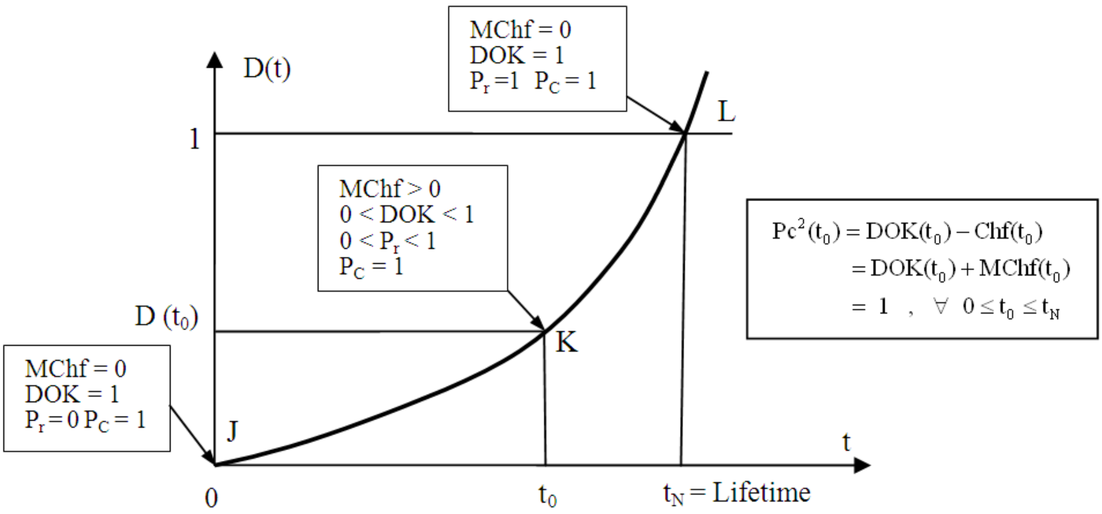

Fig. 2. EKA and the prognostic of degradation

Let us consider a degradation trajectory D(t) of a system where a specific instant t_0 is studied. The instant t_0 means here the time or age that can be measured also by the cycle number N.

Referring to the figure above (**Fig. 2**), the previous statement means that at the system age t_0, the prognostic study must give the prediction of the failure instant t_N. Therefore, the RUL predicted here at instant t_0 is the following interval: $RUL(t_0) = t_N - t_0$.

In fact, at the beginning ($t_0 = 0$) (point J), the failure probability $P_r = 0$ and the chaotic factor in our prediction is zero (Chf = 0). Therefore, $RUL(t_0 = 0) = t_N - t_0 = t_N$.

If $t_0 = t_N$ (point L) then the RUL $(t_N) = t_N - t_N = 0$ and the failure probability is one ($P_r = 1$).

If not (i.e., $0 < t_0 < t_N$) (point K), the probability of the occurrence of this instant and the prediction probability of RUL are both less than one (not certain) due to non-zero chaotic factors. The degree of our knowledge (DOK for short) is consequently less than 1. Thus, by applying here the EKA method, we can determine the system RUL with certainty in C = R + M where Pc = 1 always.

Furthermore, we need in our current study the absolute value of the chaotic factor that will give us the magnitude of the chaotic and random effects on the studied system. This new term will be denoted accordingly MChf or Magnitude of the Chaotic Factor. Hence, we can deduce the following:

$$\begin{aligned}
&\mathrm{MChf}(t_0) = |\mathrm{Chf}(t_0)| \geq 0 \text{ and} \\
&Pc^2(t_0) = \mathrm{DOK}(t_0) - \mathrm{Chf}(t_0) = \\
&\mathrm{DOK}(t_0) + |\mathrm{Chf}(t_0)| \\
&\text{since } -0.5 \leq \mathrm{Chf}(t_0) \leq 0 \\
&= \mathrm{DOK}(t_0) + \mathrm{MChf}(t_0) = 1, \quad \forall\ 0 \leq t_0 \leq t_N \\
&\Leftrightarrow 0 \leq \mathrm{MChf}(t_0) \leq 0.5 \text{ where } 0.5 \leq \mathrm{DOK}(t_0) \leq 1
\end{aligned}$$

Moreover, we can define two complementary events E and $\bar{E}$ with their respective probabilities:

$$P_{rob}(E) = p \text{ and } P_{rob}(\bar{E}) = q = 1 - p$$

Then P_{rob} (E) in terms of the instant t_0 is given by: P_{rob} (E) = P_r = P_{rob} ($t \leq t_0$) = $F(t_0)$ where F is the cumulative probability distribution function of the random variable t. Since:

$$P_{rob}(E) + P_{rob}(\bar{E}) = 1$$

Therefore:

$$\begin{aligned}
&P_{rob}(\bar{E}) = 1 - P_{rob}(E) = 1 - P_r = \\
&1 - P_{rob}(t \leq t_0) = P_{rob}(t > t_0)
\end{aligned}$$

Let us define the two particular instants: $t_0 = 0$ assumed as the initial time of functioning (raw state) corresponding to $D = D_0 = 0$ and t_N = the failure instant (wear out state) corresponding to the degradation D = 1.

The boundary conditions are.

For $t_0 = 0$ then $D = D_0$ (initial damage that may be zero or not) and:

$$F(t_0) = P_{rob}(t \leq 0) = 0$$

For $t_0 = t_N$ then D = 1 and $F(t_0) = F(t_N) = P_{rob}$ ($t \leq t_N$) = 1.

Also $F(t_0)$ is a non-decreasing function that varies between 0 and 1. In fact, $F(t_0)$ is a cumulative function (**Fig. 3**). In addition, since $RUL(t_0) = t_N - t_0$ and $0 \leq t_0 \leq t_N$ then $RUL(t_0)$ is a non-increasing remaining useful lifetime function (**Fig. 4**).

Referring to **Fig. 5** below, we can infer the following.

The complex probability $Z(t_0) = P_r(t_0) + P_m(t_0) = P_r(t_0) + i[1-P_r(t_0)]$.

The square of the norm of $Z(t_0)$ is:

$$\begin{aligned}
|Z(t_0)|^2 &= \mathrm{DOK}(t_0) = 1 + 2iP_r(t_0)P_m(t_0) \\
&= 1 - 2P_r(t_0)[1 - P_r(t_0)] \\
&= 1 - 2P_r(t_0) + 2P_r^2(t_0)
\end{aligned}$$

The Chaotic Factor and the Magnitude of the Chaotic Factor are:

$\mathrm{Chf}(t_0) = -2P_r(t_0)[1-P_r(t_0)] = -2P_r(t_0)+2P_r^2(t_0)$ is null when $P_r(t_0) = P_r(0) = 0$ (point J) or when $P_r(t_0) = P_r(t_N) = 1$ (point L) and $\mathrm{MChf}(t_0) = |\mathrm{Chf}(t_0)| = 2P_r(t_0)[1-P_r(t_0)] = 2P_r(t_0) -2P_r^2(t_0)$ is null when $P_r(t_0) = P_r(0) = 0$ (point J) or when $P_r(t_0) = P_r(t_N) = 1$ (point L)

At any instant t_0 (point K), the probability expressed in the complex set C is:

$$Pc(t_0) = P_r(t_0) + P_m(t_0)/i = P_r(t_0) + [1-P_r(t_0)] = 1 \text{ always}$$

Hence, the prediction of $RUL(t_0)$ of the system degradation in C is permanently certain.

2. APPLICATION OF EXTENDED KOLMOGOROV'S AXIOMS (EKA) TO DEGRADATION PROGNOSTIC BASED ON RELIABILITY

2.1. Review of Reliability Theory (Greene, 2004; Guillen, 1995; Gullberg, 1997; Kuhn, 1996; Liu, 2001; Mandelbrot, 1997; Montgomery and Runger, 2005; Mŭller, 2005; Orluc and Poirier, 2005; Poincaré, 1968; Prigogine and Stengers, 1992; Prigogine, 1997; Christian and Casella, 2005; Srinivasan and Mehata, 1978; Stewart, 1996; 2002; Van Kampen, 2007; Walpole, 2002; Warusfel and Ducrocq, 2004; Weinberg, 1992)

The reliability is the probabilistic evaluation of a limit state of performance on a domain of basic variables. In other words, it is obtained by the computation of the failure probability toward a criterion or a limit state.

2.1.1. Methodology

- Identify the limit states that govern the lifetime of the structure
- Identify the basic parameters intervening in the limit state
- Deduce their probability density functions
- Compute the failure probability that expresses the risk when the limit states are not satisfied

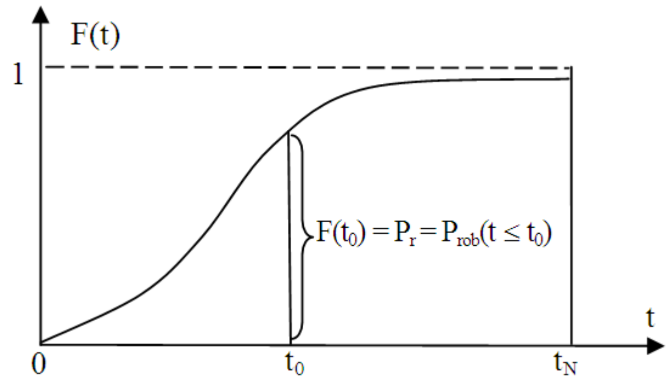

Fig. 3. Occurrence probability

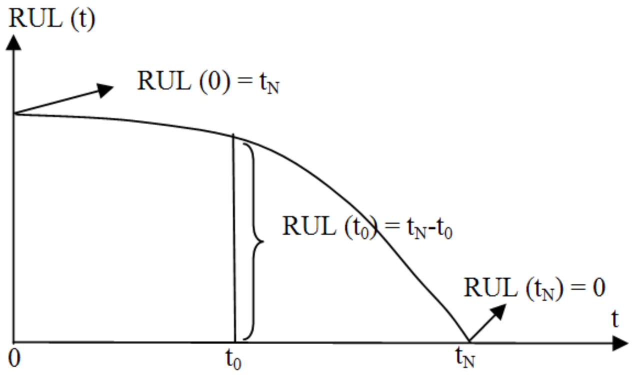

Fig. 4. RUL prognostic model

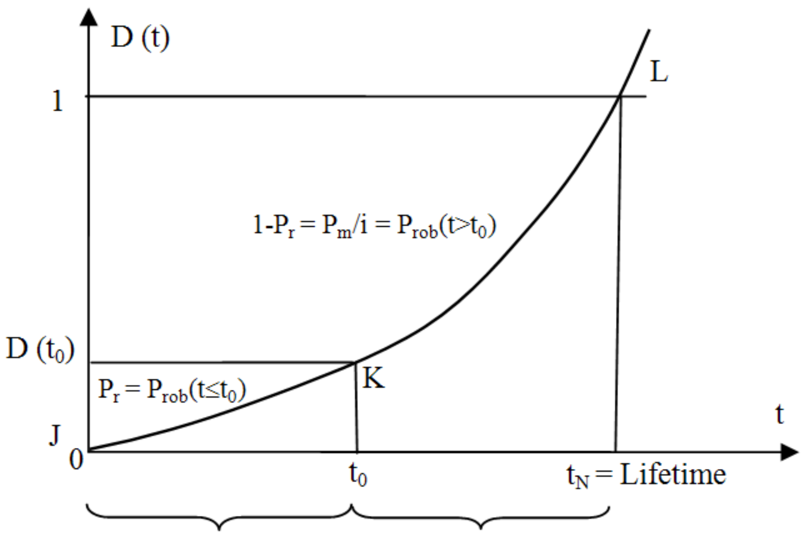

Fig. 5. Degradation prognostic model

Two types of methods exist: The Monte Carlo simulation and the approximate method First Order Reliability Method (FORM). The Monte Carlo simulation method is based on a large number of simulations and we must use N simulations when we want to evaluate a probability of order of $10^{-(N+4)}$.

The approximate method FORM is an iterative procedure that allows to calculate an index of reliability (denoted β).

The index β is the distance between the origin and the limit state function G(t) in a standard space. Once we have calculated β we can deduce the failure probability $P_r = \Phi(-\beta)$.

In FORM approximation the real (usually nonlinear) limit state is replaced by its tangent plane at a specific point called the Most Probable Failure Point (MPFP). This point is the closest point on G(t) to the origin.

The limit state G(t) divides the space into two regions:

- First region where $G(t) > 0$ called safe region
- And the second region where $G(t) \leq 0$ called failure region

2.1.2. Work Plan

We choose, in a general case, N random variables, correlated and of any density functions, as well as a nonlinear limit state function. This method is based on the following iterative algorithm:

- Transforming of basic random variables into standard normal variables N(0,1)
- Transforming the limit state from the original space to the standard normal space
- Search of the MPFP point by replacing the limit state surface by its tangent hyper-plane at the same point
- Calculate the index β and the probability of failure P_f

2.1.3. Description of the Algorithm

The transformation from the basic state to the normalized state is implicit in the algorithm. The steps are the following (**Fig. 6**):

Let the limit state equation be: g(z)

where, z = z1,z2,z3,…..zn is the random vector of the limit state, therefore.

1) Initialization of the coordinates of MPFP. The mean value of each variable is a good choice:

$$Z^1 = \mu_{z1}, \mu_{z2}, \ldots\ldots, \mu_{zn}$$

2) Calculate the following parameters: (m is the number of the iteration).

The value of the limit state at MPFP:

$$g_0^m = g\left(z_1^m, \ldots., z_2^m\right)$$

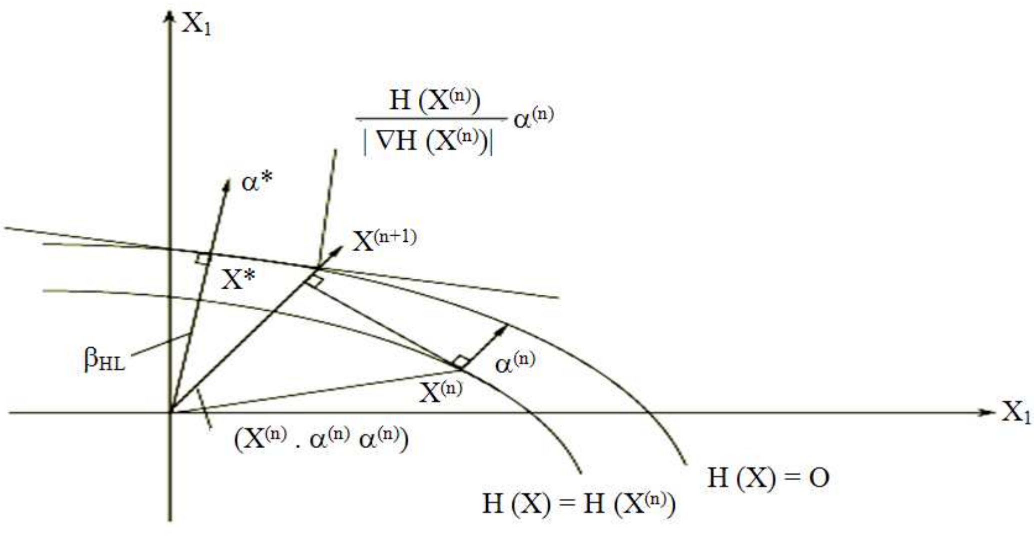

Fig. 6. The First Order Reliability Method (FORM)

The gradient at MPFP is assumed to be:

$$g_i^m = \frac{\partial g}{\partial zi}\left(z_1^m, \ldots, z_2^m\right)$$

The equivalent normal standard deviation and mean value of non-normal variables:

$$\sigma_i^m = \frac{\varphi(\varnothing^{-1}(F_{zi}(z_i^m))}{f_{zi}(z_i^m)}$$

$$\mu_i^m = z_i^m - \sigma_i^m \varnothing^{-1}\left(F_{zi}\left(z_i^m\right)\right)$$

3) Calculate the intermediate parameters:

$$z^m = \sum_{i=1}^{n} g_i^{'m} z_i^m$$

$$\mu_z^m = \sum_{i=1}^{n} g_i^{'m} \mu_i^m$$

$$\sigma_z^m = \sqrt{\sum_{i=1}^{n} \left(g_i^{'m}\right)^2 \left(\sigma_i^m\right)^2}$$

4) Calculate:
The directive cosine:

$$\alpha_i = -\frac{g_i^{'m}\sigma_i^m}{\sigma_z^m}$$

The reliability index:

$$\beta^m = -\frac{z^m - g_0^m - \mu_z^m}{\sigma_z^m}$$

The new coordinates of MPFP:

$$z_i^m = \mu_i^M + \alpha_i^m \beta^m \sigma_i^m$$

5) Verify the convergence criterion:

$$\left\|z^{m+1} - z^m\right\| \le \text{to}\,l \text{ and } \left|\beta^{m+1} - \beta^m\right| \le \text{to}\,l$$

6) Repeat the steps from 2 till 5 until convergence.
7) Calculate the failure probability:

$$Pf = \varnothing(-\beta)$$

2.2. Application of FORM to Prognostic

In this part, we study the extended Kolmogorov axioms in the context of reliability by defining a limit state G that describes the lifetime margin of the system. For each value of an instant t_0 we determine its corresponding probability of survival or of the Remaining Useful Lifetime (RUL).

We have:

$$G(t_0) = RUL(t_0) = t_N - t_0$$

Where:

$G(t_0)$ = The limit state of lifetime.

t_N = The fixed lifetime of the system which follows a normal distribution N(0.0006 $\overline{t}_N$; 1)

t_0 = An arbitrary instant that varies from 0 to t_N and which follows a normal distribution $N\left(\overline{t_0}; 0.1 \times \overline{t_0}\right)$

When RUL (t_0) is zero or negative then we have a case of $t_0 \ge t_N$ that means that we have a system failure that cannot live until the instant t_0. In the other case where $t_0 < t_N$, the system can live above the instant t_0 and we have a case of success.

The probability:

$$P_r(t_0) = P_{rob}\{G(t_0) \le 0\} = P_{rob}\{RUL(t_0) \le 0\}$$

is computed by the FORM (First Order Reliability Method) procedure that uses a reliability index β.

$\beta = -\Phi^{-1}[P_r(t_0)]$ where P_r (t_0) is the cumulative probability and Φ is the normal cumulative distribution function. Hence, Φ^{-1} is the inverse of Φ and $P_r(t_0) = \Phi(-\beta)$.

In the extended Kolmogorov's axioms, the real part of probability is taken here P_r (t_0). As we make the instant t_0 vary between 0 and t_N, then P_r (t_0) varies between 0 and 1 (**Fig. 7**).

Knowing that we take t_0 and t_N as two normal random variables where the value of t_N corresponds to nearly 5798 number of cycles (critical value: N_c). After a reliability calculation using a Matlab program, we deduce a value of P_r (t_0) for each value of instant t_0. For this set of P_r (t_0) we have computed and ploted the extended Kolmogorov's parameters and components $Chf(t_0)$, $MChf(t_0)$, $DOK(t_0)$, $Pc(t_0)$, $P_m(t_0)/i$.

Therefore, we get the following figures (**Fig. 8 and 9**).

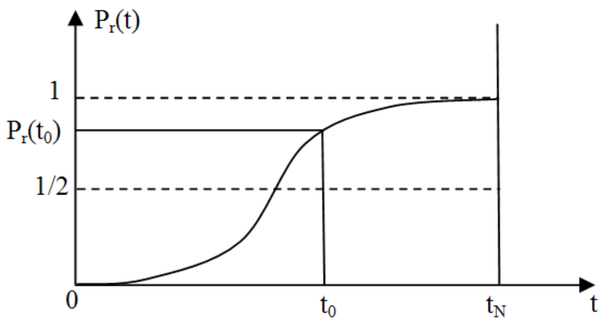

Fig. 7. Probability of failure

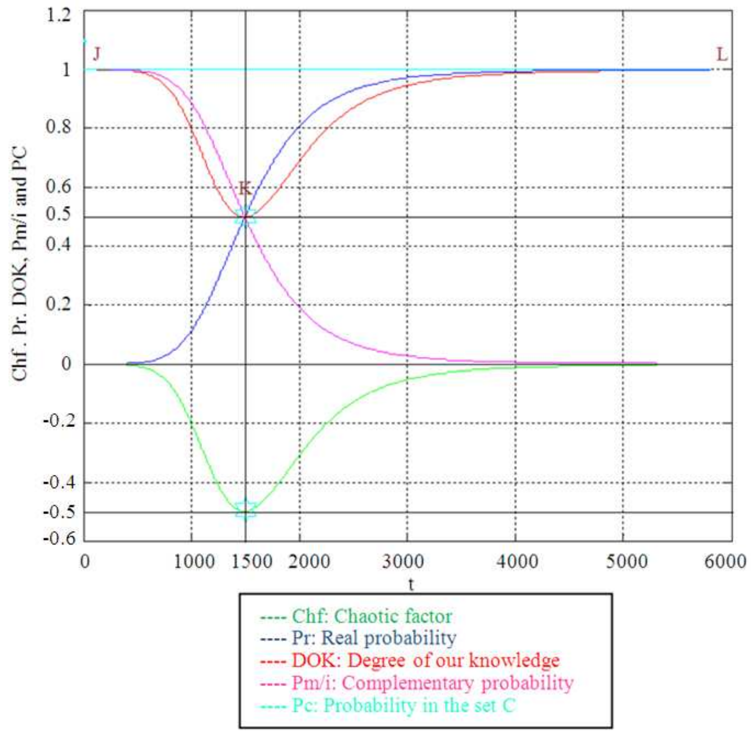

Fig. 8. DOK, Chf and Pc as functions of the probability of failure

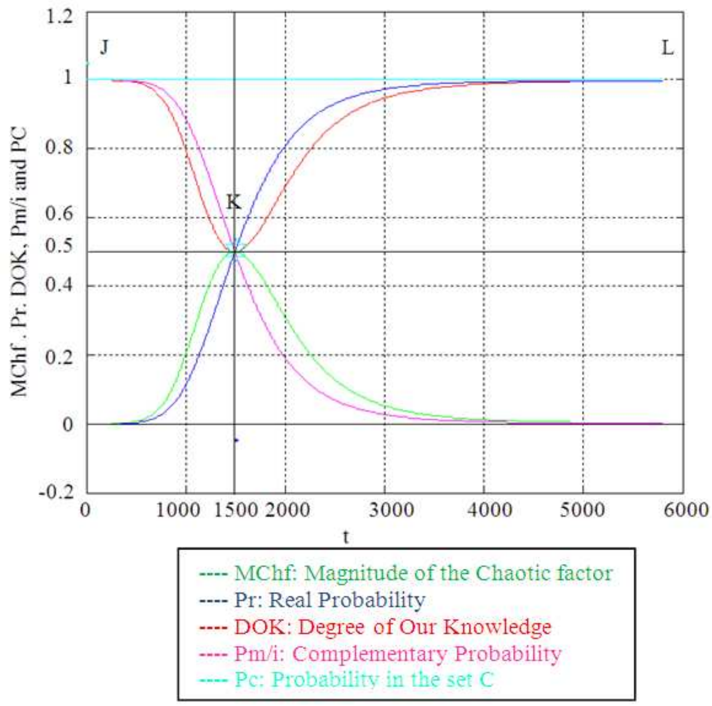

Fig. 9. DOK, MChf and Pc as functions of the probability of failure

We note from the figure that the DOK is maximum (DOK = 1) when absolute value of Chf which is MChf is minimum (MChf = 0) (points J & L), that means when the magnitude of the chaotic factor (MChf) decreases our certain knowledge (DOK) increases. Afterward, MChf starts to increase during the functioning due to the environment and intrinsic conditions thus leading to a decrease in DOK until they both reach 0.5 at t_0 = 1500 (point K). The real probability P_r and the complementary probability P_m/i will intersect with DOK also at the point (1500, 0.5) (point K). With the increase of t_0, the Chf and MChf return to zero and the DOK returns to 1 where we reach total damage (D = 1) and hence the total certain failure (P_r = 1) of the system (point L). At this last point the failure here is definite, $P_r(t_N) = 1$ and $RUL(t_N) = t_N - t_N = 0$ with $Pc(t_N) = 1$, so the logical explanation of the value DOK = 1 follows.

We note that the point K is not at the middle of DOK since the probability of failure distribution evaluated by FORM is not symmetric.

Furthermore, at each instant t_0, the remaining useful lifetime $RUL(t_0)$ is certainly predicted in the complex set C with Pc maintained as equal to one through continuous compensation between DOK and Chf. This compensation is from instant $t_0 = 0$ where $D(t_0) = 0$ until the failure instant t_N where $D(t_N) = 1$.

2.2.1. The Cube of Probability Components

In the following figure, we represent the extended Kolmogorov's probability components P_r and P_m/i in a three dimensional graph in terms of t and of each other (**Fig. 10**).

It is important to mention that if we rescale the time axis to an interval [0,1] so the minimal value of DOK is at the instant $t_0 = \frac{1500}{5798} = 0.2587$ where N = 5798 cycles corresponds to t_N. This last important point is clearly shown in this cube and in the following one.

From the cube below, we can notice that the probability Pc in the complex set C = R + M is obtained at each instant t_0 as the sum of P_r and P_m/i and is always equal to one.

2.2.2. The Cube of Probability Parameters

In the following figure, we represent the extended Kolmogorov's probability parameters DOK and Chf in a three dimensional graph in terms of t and of each other (**Fig. 11**).

2.3. Example: Application to Young Modulus

We consider once again the Young modulus example previously treated in the first paper on extended kolmogorov's axioms and "Complex Probability Theory".

Let E be the Young modulus in a material bar domain (**Fig. 12**) and we assume that it follows a Normal Gaussian distribution.

The limit state considered here for FORM analysis is:

$$G(E_0) = RUL(E_0) = E_N - E_0$$

When RUL (E_0) is zero or negative then we have the case of $E_0 \geq E_N$ that means that we have a system failure that cannot live until E_0. In the other case where $E_0<E_N$, the system can live above E_0 and we have a case of success.

The real Probability of failure is given by:

$$\begin{aligned} P_r(E_0) &= P_{rob}\{G(E_0) \leq 0\} = P_{rob}\{RUL(E_0) \leq 0\} \\ &= P_{rob}\{E_N \leq E_0\} \end{aligned}$$

The reliability index $\beta = -\Phi^{-1}(P_r)$.

In the extended Kolmogorov's axioms, the real part of probability is taken here as P_r (E_0). As we make the E_0 vary between 0 and E_N, then P_r (E_0) varies between 0 and 1 (**Fig. 13**).

Let $\overline{E}$ be the mean value of E and is taken to be equal to 29575 Ksi. Let σ_E be the standard deviation of E and is equal to 1507 Ksi. Let the coefficient of variation be $c.v = \frac{\sigma_E}{\overline{E}} = \frac{1507}{29575} = 0.050955198 \cong 0.051$.

Take $E_0 = \overline{E}$ =29575 Ksi.

We can compute from the statistical tables that:

$P_{rob}[-\infty < E \leq 29575 \text{ Ksi}] = 0.5$ and $P_{rob}[29575 \text{ Ksi} \leq E < +\infty] = 0.5$.

As well $P_{rob}[E \leq 0] \cong 0$.

Note that:

$$\Phi(u_0) = \int_{-\infty}^{u_0} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right).du = P_{rob}[u \leq u_0]$$

Where:

$$u = \frac{E - \overline{E}}{\sigma_E}$$

In the real domain R we have:

$$dF = f_E(u).du = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right).du$$

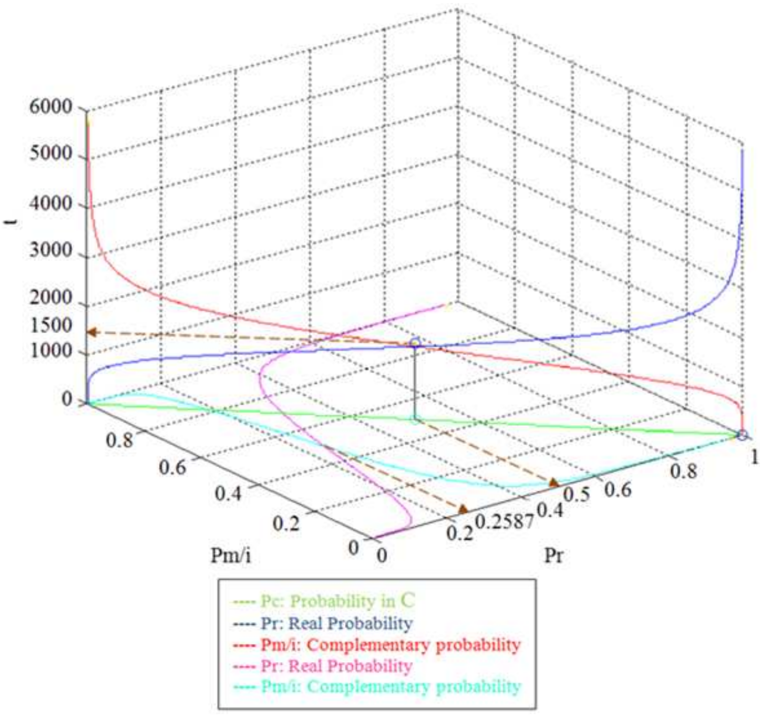

Fig. 10. The probabilities P_r and P_m/i in terms of t and of each other

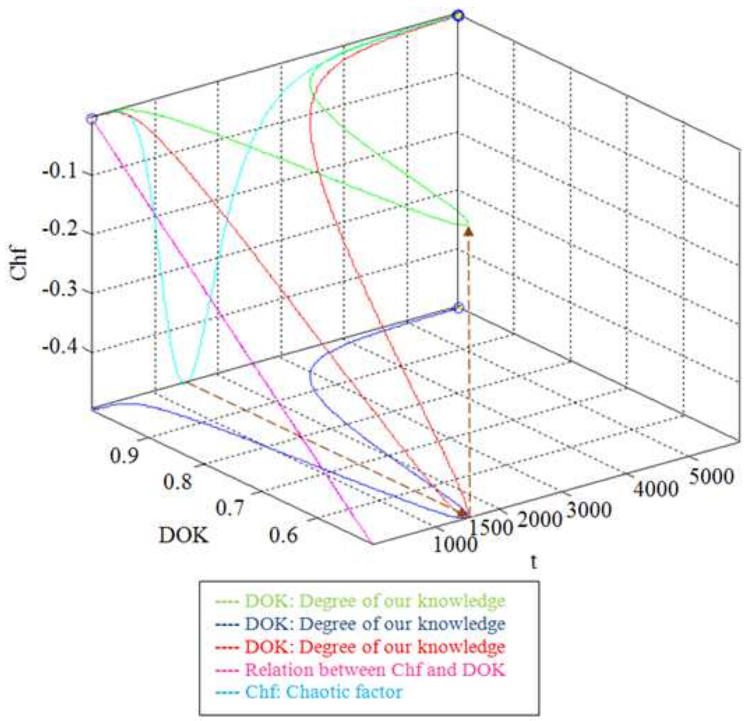

Fig. 11. DOK and Chf in terms of t and of each other

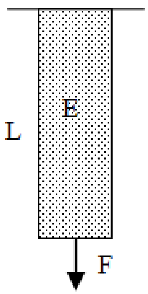

Fig. 12. The Young modulus E in a material domain

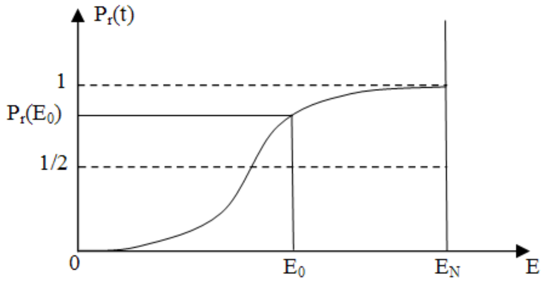

Fig. 13. Probability of failure

And:

$$\int_{-\infty}^{+\infty} dF = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du$$
$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_E} \exp\left[-\frac{1}{2}\left(\frac{E-\overline{E}}{\sigma_E}\right)^2\right] dE = 1$$

Now:

$$P_{rob}[-\infty < E \le 29575] = F(29575)$$
$$= Pr_0 = \int_{-\infty}^{29575} \frac{1}{\sqrt{2\pi}.1507} \exp\left[-\frac{1}{2}\left(\frac{E-29575}{1507}\right)^2\right].dE$$
$$= 0.5$$

The correspondent probability in the imaginary domain M is:

$$Pm_0 = i(1-Pr_0) = i.P_{rob}[E > 29575] = i.[1-F(29575)]$$
$$= i.\int_{29575}^{+\infty} \frac{1}{\sqrt{2\pi}1507} \exp\left[-\frac{1}{2}\left(\frac{E-29575}{1507}\right)^2\right].dE$$
$$= i \times 0.5$$

If we compute the norm of the complex number:

$$Z_0 = Pr_0 + Pm_0$$

we have:

$$|Z_0|^2 = Pr_0^2 + (Pm_0/i)^2 = Pr_0^2 + (1-Pr_0)^2$$
$$= 1 + 2Pr_0(Pr_0 - 1) = 1 - 2Pr_0(1-Pr_0);$$

This implies that:

$$1 = |Z_0|^2 + 2Pr_0(1-Pr_0)$$
$$= |Z_0|^2 - 2.i^2.Pr_0.(1-Pr_0)$$
$$= |Z_0|^2 - 2.i.Pr_0.Pm_0$$
$$= Pr_0^2 + (Pm_0/i)^2 - 2.i.Pr_0.Pm_0$$
$$= (Pr_0 + Pm_0/i)^2 = Pc_0^2 \Rightarrow Pc_0 = 1$$

We note that:

$$Z_0 = Pr_0 + Pm_0 = \int_{-\infty}^{E_0} f_E(u)du + i\int_{E_0}^{+\infty} f_E(u)du = 0.5 + i \times 0.5$$

We have also:

$$Pc_0^2 = (Pr_0 + Pm_0/i)^2 = \left(\int_{-\infty}^{E_0} + \int_{E_0}^{+\infty}\right)^2 = \left(\int_{-\infty}^{+\infty}\right)^2 = 1^2 = 1$$

and the chaotic factor is:

$$Chf_0 = 2.i.Pr_0.Pm_0 = 2.i \times \int_{-\infty}^{E_0} \times i \times \int_{E_0}^{+\infty}$$
$$= -2 \times \int_{-\infty}^{E_0} \times \left(1 - \int_{-\infty}^{E_0}\right)$$

Where:

$$Chf_0 = 0 \text{ if } \begin{cases} E_0 \to -\infty, & \text{hence } Pr_0 = 0 \\ E_0 \to +\infty, & \text{hence } Pr_0 = 1 \end{cases}$$

Moreover, the Magnitude of the chaotic factor is:

$$MChf_0 = |2.i.Pr_0.Pm_0| =$$
$$\left|2.i \times \int_{-\infty}^{E_0} \times i \times \int_{E_0}^{+\infty}\right| = \left|-2 \times \int_{-\infty}^{E_0} \times \left(1 - \int_{-\infty}^{E_0}\right)\right| = 2 \times \int_{-\infty}^{E_0} \times \left(1 - \int_{-\infty}^{E_0}\right)$$

Where:

$$\mathrm{MChf}_0 = 0 \text{ if } \begin{cases} E_0 \to -\infty, & \text{hence } \Pr_0 = 0 \\ E_0 \to +\infty, & \text{hence } \Pr_0 = 1 \end{cases}$$

Therefore, we say that:

$Pc_0{}^2 = |Z_0|^2 - 2.i.Pr_0.Pm_0 =$ Degree of our knowledge-Chaotic factor = 1

and if $Chf_0 = 0 \Rightarrow |Z_0|^2 = 1$ in other words, if the chaotic factor is zero, then the degree of our knowledge is 1 or 100%.

In addition, we say that:

$Pc_0{}^2 =$ Degree of our knowledge + Magnitude of the Chaotic factor = 1.

and if $MChf_0 = 0 \Rightarrow |Z_0|^2 = 1$, in other words, if the magnitude of the chaotic factor is zero, then the degree of our knowledge is 1 or 100%.

Numerically, we write:

$$|Z_0|^2 = (0.5)^2 + (0.5)^2 = 0.25 + 0.25 = 0.5$$
$$\Rightarrow |Z_0| = 0.707107 \Rightarrow Chf_0 \neq 0, \quad \text{Notice that } \frac{1}{2} \leq |Z_0|^2 \leq 1$$

Hence:

$$Chf_0 = 0.5 - 1 = -0.5, \text{Notice that } -\frac{1}{2} \leq Chf_0 \leq 0$$

And:

$$MChf_0 = |Chf_0| = |-0.5| = 0.5, \text{Notice that } 0 \leq MChf_0 \leq \frac{1}{2}$$

Consequently, we can say that.

The degree of our knowledge DOK= $|Z_0|^2 = 0.5$, the chaotic factor Chf_0 =-0.5 and the magnitude of the chaotic factor $MChf_0 = 0.5$.

What is interesting here is thus we have quantified both the degree of our knowledge and the chaotic factor of the event as well as the correspondent magnitude of the chaotic factor.

Notice that:

$$\text{DOK-Chf} = 0.5 - (-0.5) = 0.5 + 0.5 = 1 = Pc_0$$

$$\text{DOK + MChf} = 0.5 + 0.5 = 1 = Pc_0$$

Conversely, if we assume that:

$$Chf_0 = 0 \Rightarrow MChf_0 = 0 \Rightarrow |Z_0|^2 = 1 \Rightarrow Pr_0^2 + (Pm_0/i)^2 = 1$$
$$\text{If } Chf_0 = -\frac{1}{2} \Rightarrow MChf_0 = \frac{1}{2} \Rightarrow E_0 = \overline{E} \text{ and } |Z_0|^2 = \frac{1}{2}$$
$$\Rightarrow 2Pr_0(1 - Pr_0) = 0 \Rightarrow \begin{cases} Pr_0 = 0 \\ \text{or} \\ Pr_0 = 1 \end{cases} \Rightarrow \begin{cases} E_0 \to -\infty \\ \text{or} \\ E_0 \to +\infty \end{cases}$$

If E_0 increases to become=4000 then both $|Z_0|^2$ and Chf_0 increase and $MChf_0$ decreases.

Therefore we can infer that:

$$\lim_{E_0 \to +\infty} (Chf_0) = 0, \lim_{E_0 \to +\infty} (MChf_0) = 0 \text{ and } \lim_{E_0 \to +\infty} \left(|Z_0|^2\right) = 1$$

Where:

$$\begin{aligned} Pc_0{}^2 &= |Z_0|^2 - Chf_0 \\ &= DOK_0 - Chf_0 \\ &= |Z_0|^2 + MChf_0 \\ &= DOK_0 + MChf_0 = 1, \end{aligned}$$

for every E_0 in the real set R.

We note from the figure below that the DOK is maximum (DOK = 1) when absolute value of Chf which is MChf is minimum (MChf = 0) (points J & L), that means when the magnitude of the chaotic factor (MChf) decreases, our certain knowledge (DOK) increases. Afterward, MChf starts to increase during the functioning due to the environment and intrinsic conditions thus leading to a decrease in DOK until they both reach 0.5 at E_0 = 29575 (point K). The real probability P_r and the complementary probability P_m/i will intersect with DOK also at the point (29575, 0.5) (point K). With the increase of E_0, the Chf and MChf return to zero and the DOK returns to 1 where we reach total damage (D = 1) and hence the total certain failure (P_r = 1) of the system (point L). At this last point the failure here is definite, $P_r(E_N) = 1$ and $RUL(E_N) = E_N - E_N = 0$ with $Pc(E_N) = 1$, so the logical explanation of the value DOK = 1 follows (**Fig. 14 and 15**).

We note that the point K is not at the middle of DOK since the probability of failure distribution evaluated by FORM is not symmetric.

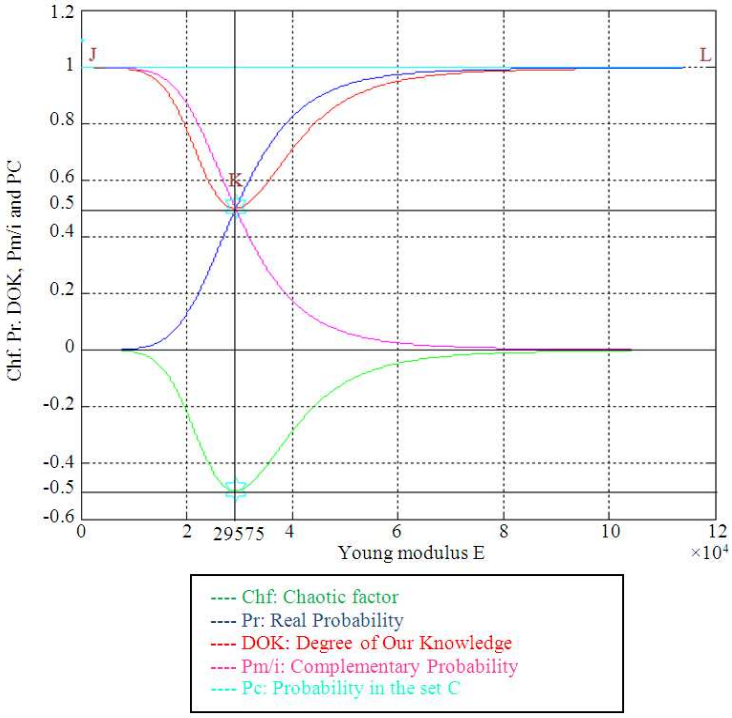

Fig. 14. DOK, Chf and Pc as functions of the probability of failure

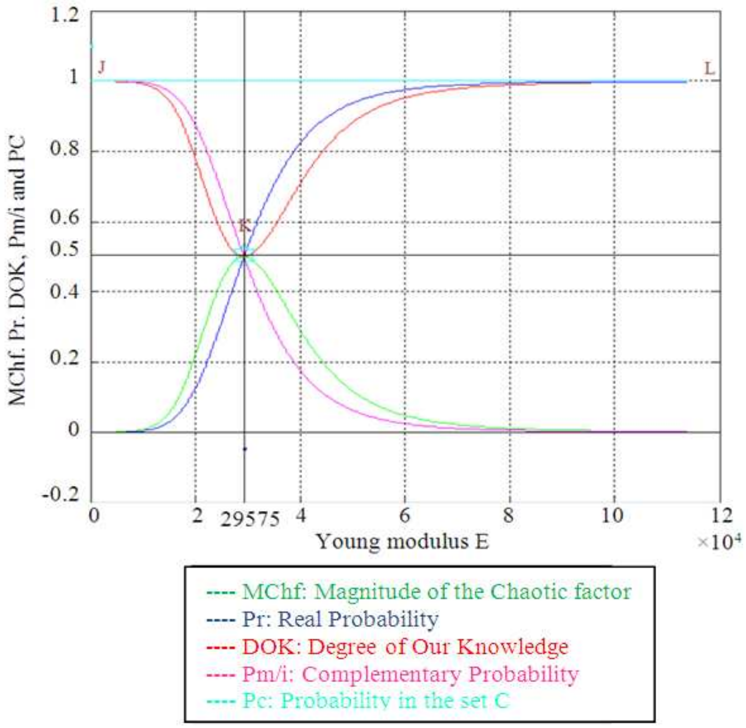

Fig. 15. DOK, MChf and Pc as functions of the probability of failure

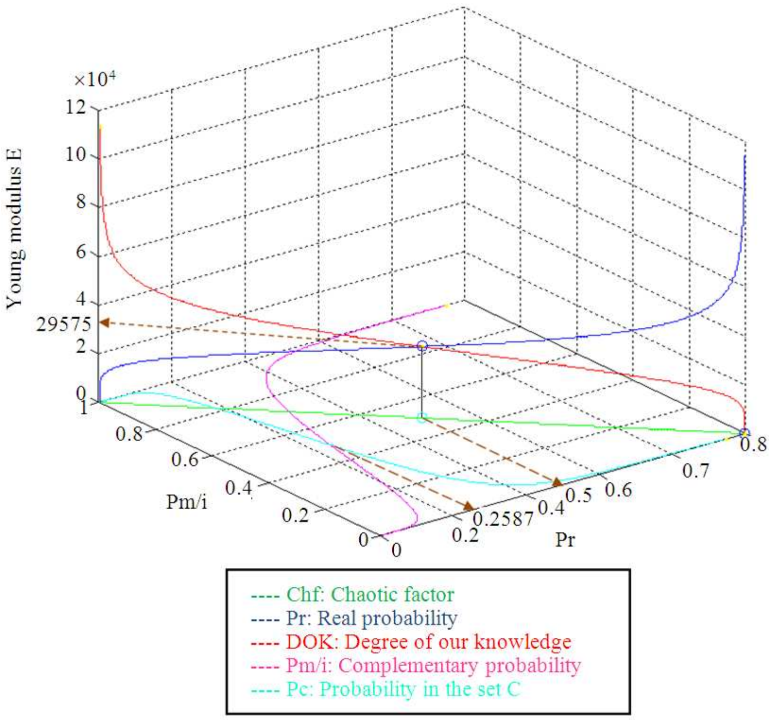

Fig. 16. The probabilities P_r and P_m/i in terms of E and of each other

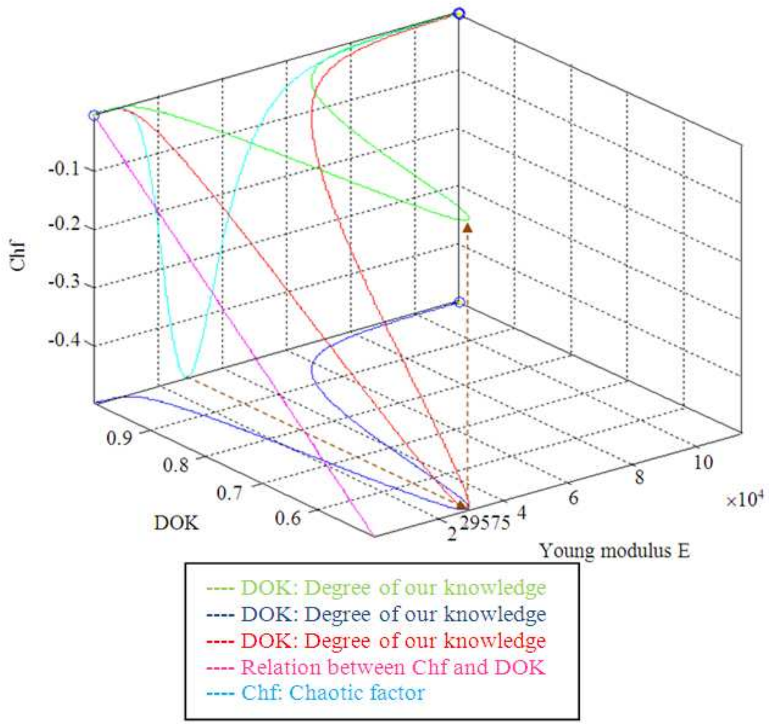

Fig. 17. DOK and Chf in terms of E and of each other

Furthermore, at each E_0, the remaining useful lifetime $RUL(E_0)$ is certainly predicted in the complex set C with Pc maintained as equal to one through a continuous compensation between DOK and Chf. This compensation is from $E_0 = 0$ where $D(E_0) = 0$ until failure at E_N where $D(E_N) = 1$.

2.3.1. The Cube of Probability Components

In the figure above, we represent the extended Kolmogorov's probability components P_r and P_m/i in a three dimensional graph in terms of E and of each other (**Fig. 16**).

It is important to mention that if we rescale the E axis to an interval [0,1] so the minimal value of DOK is at the instant where $E_0 = \frac{29575}{114317} = 0.2587$ knowing that E = 114317 corresponds to E_N. This last important point is clearly shown in this cube and in the following one.

From the cube above, we can notice that the probability Pc in complex set C = R + M is obtained at each value E_0 as the sum of P_r and P_m/i and is always equal to one.

2.3.2. The Cube of Probability Parameters

In the figure above, we represent the extended Kolmogorov's probability parameters DOK and Chf in a three dimensional graph in terms of E and of each other (**Fig. 17**).

3. CONCLUSION

In this study I applied the theory of Extended Kolmogorov Axioms to Prognostic based on Reliability. I used for this purpose the very well known First Order Reliability Method or FORM analysis for short. Consequently, I established a tight link between the new theory and degradation or the remaining useful lifetime and reliability. Hence, I developed the theory of "Complex Probability" beyond the scope of the previous three papers on this topic. As it was proved and illustrated, when the degradation index is 0 or 1 and correspondingly the RUL is t_N or 0 then the Degree of Our Knowledge (DOK) is one and the chaotic factor (Chf and MChf) is 0 since the state of the system is totally known. During the process of degradation ($0<D<1$) we have: $0.5<DOK<1$, $-0.5<Chf<0$ and $0<MChf<0.5$. Notice that during the whole process of degradation we have Pc = DOK - Chf = DOK + MChf = 1, that means that the phenomenon which seems to be random and stochastic in R is now deterministic and certain in C = R + M and this after adding to R the contributions of M and hence after subtracting the chaotic factor from the degree of our knowledge. Moreover, for each value of an instant t_0 or E_0, I have determined their corresponding probability of survival or of the remaining useful lifetime $RUL(t_0) = t_N - t_0$ or $RUL(E_0) = E_N - E_0$. In other words, at each instant t_0 or E_0, $RUL(t_0)$ or $RUL(E_0)$ are certainly predicted in the complex set C with Pc maintained as equal to one through a continuous compensation between DOK and Chf. This compensation is from instant $t_0 = 0$ where $D(t_0) = 0$ until the failure instant t_N where $D(t_N)=1$ and this compensation is also from $E_0 = 0$ where $D(E_0) = 0$ until failure at E_N where $D(E_N)=1$. Furthermore, using all these graphs illustrated throughout the whole paper, we can visualize and quantify both the system chaos (Chf and MChf) and the system certain knowledge (DOK and Pc). Additionally, an application to Young modulus was successfully done here. This is certainly very interesting and fruitful and shows once again the benefits of extending Kolmogorov's axioms and thus the originality and usefulness of this new field in mathematics that can be called verily: "The Complex Probability and Statistics Paradigm".

4. REFERENCES

Abou Jaoude, A., 2005. Computer simulation of monte carlo methods and random phenomena. Ph.D. Thesis, Bircham International University.

Abou Jaoude, A., 2007. Analysis and algorithms for the statistical and stochastic paradigm. Ph.D. Thesis, Bircham International University.

Abou Jaoude, A., 2012. Advanced Analytical Model for the Prognostic of Industrial Systems Subject to Fatigue. Ph.D. Thesis, EDST-The Lebanese University and Aix-Marseille University-France.

Abou Jaoude, A., 2013a. The complex statistics paradigm and the law of large numbers. J. Math. Stat., 9: 289-304. 10.3844/jmssp.2013.289.304

Abou Jaoude, A., 2013b. Complex probability theory and prognostic. J. Math. Stat., 9: 305-328. 10.3844/jmssp.2013.305.328

Abou Jaoude, A., K. El-Tawil and S. Kadry, 2010. Prediction in complex dimension using kolmogorov's set of axioms. J. Math. Stat., 6: 116-124. 10.3844/jmssp.2010.116.124

Bell, E.T., 1992. The Development of Mathematics. 1st Edn., Courier Dover Publications, New York, ISBN-10: 0486152286, pp: 637.

Benton, W., 1996. Encyclopedia Britannica. 1st Edn., Encyclopedia Britannica Inc., Chicago.

Boursin, J.L., 1986. Les Structures du Hasard. 1st Edn., Seuil, ISBN-10: 2020092352, pp: 186.

Chan Man Fong, C.F., D. De Kee and P.N. Kaloni, 1997. Advanced Mathematics for Applied and Pure Sciences. 1st Edn., Gordon and Breach, Amsterdam, ISBN-10: 9056996142, pp: 880.

Cheney, E.W. and D.R. Kincaid, 2004. Numerical Mathematics and Computing. 5th Edn., Thomson-Brooks/Cole, Belmont, ISBN-10: 0534389937, pp: 817.

Christian, R. and G. Casella, 2005. Monte Carlo Statistical Methods. 2nd Edn., Springer, ISBN-10: 0387212396, pp: 649.

Dacunha-Castelle, D., 1996. Chemins de L'aleatoire: Le Hasard et le Risque Dans la Societe Moderne. 1st Edn., Flammarion, ISBN-10: 2082112268, pp: 265.

Dalmedico Dahan, A. and J. Peiffer, 1986. Une Histoire des Mathématiques: Routes et Dédales. 2nd Edn., Edition du Seuil, ISBN-10: 2020091380, pp: 308.

Dalmedico Dahan, A., J.L. Chabert and K. Chemla, 1992. Chaos et Déterminisme. 1st Edn., Edition du Seuil, Paris, ISBN-10: 2020151820, pp: 414.

Ekeland, I., 1991. Au Hasard: La Chance, la Science, et le Monde. 1st Edn., Editions du Seuil, Paris, ISBN-10: 2020128772, pp: 198.

Feller, W., 1968. An Introduction to Probability Theory and its Applications. 3rd Edn., Wiley, New York.

Finney, Weir and Giordano, 2004. Thomas' Calculus. 10th Edn., Addison Wesley Longman. United States of America.

Gentle, J.E., 2003. Random Number Generation and Monte Carlo Methods. 1st Edn., Springer, New York, ISBN-10: 0387001786, pp: 381.

Gerald, F.C. and O.P. Wheatley, 1999. Applied Numerical Analysis. 6th Edn., Addison Wesley, Reading, ISBN-10: 0201474352, pp: 698.

Gleick, J., 1997. Chaos: Making a New Science. 1st Edn., Vintage, London, ISBN-10: 0749386061, pp: 352.

Greene, B., 2000. The Elegant Universe. 1st Edn., Demco Media, ISBN-10: 0606252657, pp: 447.

Greene, B., 2004. The Fabric of the Cosmos: Space, Time and the Texture of Reality. 1st Edn., Allen Lane, Penguin Books, ISBN-10: 0713996773, pp: 569.

Guillen, M., 1995. Initiation aux Mathématiques. 1st Edn., Albin Michel.

Gullberg, J., 1997. Mathematics: From the Birth of Numbers. 1st Edn., W.W. Norton and Company, New York, ISBN-10: 039304002X, pp: 1093.

Kuhn, S.T., 1996. The Structure of Scientific Revolutions. 3rd Edn., University of Chicago Press, Chicago, ISBN-10: 0226458083, pp: 212.

Liu, J.S., 2001. Monte Carlo Strategies in Scientific Computing. 1st Edn., Springer, New York, ISBN-10: 0387952306, pp: 343.

Mandelbrot, B., 1997. Fractales, Hasard et Finances. 1st Edn., Flammarion, Paris.

Montgomery, C.D. and C.G. Runger, 2005. Applied Statistics and Probability for Engineers. 3rd Edn., John Wiley and Sons, New York, ISBN-10: 0471735566, pp: 720.

Mũller, X., 2005. Mathématiques: l'ordinateur aura bientot le dernier mot. Science et Vie.

Orluc, L. and H. Poirier, 2005. Génération fractale, les enfants de mandelbrot. Science et Vie.

Poincaré, H., 1968. La science et l'Hypothèse. 1st Edn., Flammarion, Paris, pp: 252.

Prigogine, I. and I. Stengers, 1992. Entre le Temps et l'Eternité. 1st Edn., Flammarion, Paris, ISBN-10: 2080812629, pp: 222.

Prigogine, I., 1997. The End of Certainty. 1st Edn., Free Press, New York, ISBN-10: 0684837056, pp: 228.

Srinivasan, S.K. and K.M. Mehata, 1978. Stochastic Processes. 1st Edn., McGraw-Hill, New Delhi, ISBN-10: 0070966125, pp: 388.

Stewart, I., 1996. From Here to Infinity. 1st Edn., Oxford University Press, New York, ISBN-10: 0192832026, pp: 310.

Stewart, I., 2002. Does God Play Dice?: The New Mathematics of Chaos. 2nd Edn., Blackwell Publishing, Oxford, ISBN-10: 0631232516, pp: 416.

Van Kampen, N.G., 2007. Stochastic Processes in Physics and Chemistry. 3rd Edn., North Holland, ISBN-10: 0444529659, pp: 464.

Walpole, R.E., 2002. Probability and Statistics for Engineers and Scientists. 7th Edn., Prentice Hall, Upper Saddle River, ISBN-10: 0130415294, pp: 730.

Warusfel, A. and A. Ducrocq, 2004. Les Mathématiques, Plaisir et Nécéssité. 1st Edn., Editions Vuibert.

Weinberg, S., 1992. Dreams of a Final Theory. 1st Edn., Pantheon, New York, ISBN-10: 0679419233, pp: 334.