Original Research Paper

Estimation for Unknown Parameters of the Burr Type-XII Distribution Based on an Adaptive Progressive Type-II Censoring Scheme

[1,2]**Montaser M. Amein**

[1]*Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt*
[2]*Department of Mathematics and Statistics, Faculty of Science, Taif University, Hawia (888), Taif, Kingdom of Saudi Arabia*

Abstract: In this study, the Maximum Likelihood Estimation (MLE) and Bayes estimation are exploited to make interval estimation based on adaptive progressive Type-II censoring for the Burr Type-XII distribution. Explicit form for the parameters of Bayes estimator doesn't exist, so, Markov Chain Monte Carlo (MCMC) method is used as approximation to find posterior mean under squared error loss function. Real data set are presented to illustrate the methods of inference.

Keywords: An Adaptive Type-II Progressive Censoring Scheme, Bayesian and Non-Bayesian Estimations, Gibbs and Metropolis Sampler, Burr Type-XII Distribution

Introduction

Censored sampling have been attracting great interest due to their wide applications. In life testing, the experimenter may not always obtain complete information on failure times for all experimental units. Reducing the total test time and the associated cost is one of the major reasons for censoring. A more general censoring scheme called progressive Type-II right censoring, in which n independent items are put on life test with continuous, identically distributed failure times $X_1, X_2, \cdots, X_n$, with a censoring scheme $(R_1, R_2, ..., R_m)$ is prefixed (Fig. 1). Progressive censoring have studied by many authors. Among them are (Viveros and Balakrishnan, 1994; Aggarwala and Balakrishnan, 1996; Balakrishnan and Aggarwala, 2000; Wu, 2002; Balakrishnan *et al.*, 2002; Balakrishnan, 2007; Ng and Chan, 2007). For extensive survey of progressive censored, see (Balakrishnan and Aggarwala, 2000).

The Model Description

Recently, Ng *et al.* (2010) have established an adaptive Type-II progressive censoring scheme. In which, a mixture of order statistics and Type-II progressive censoring schemes. For this censoring scheme, time T and the number of observed failures m is prefixed and $T > m$. Suppose J is the number of failures observed before time T, i.e., $X_{J:m:n} < T < X_{J+1:m:n}$, $J = 0, 1, 2, \cdots, m$ where $X_{0:m:n} = 0$, $X_{m+1:m:n} = \infty$. After the

experiment passed time T, set $R_{j+1} = \cdots = R_m = 0$. Figure 2 gives the schematic representation of this situation.

Two cases for the value of T differ from the adaptive Type-II progressive censoring. First: ($m < T$): The conventional a usual progressive Type-II censoring scheme with ($R_1, R_2, ..., R_m$). Second: Extreme case can occur when $T = 0$, which results in the conventional Type-II censoring scheme $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n - m$. Many papers dealing with adaptive progressive Type-II censoring have been appeared. Among them, Soliman *et al.* (2011; 2012) have used a Bayes study using MCMC approach based on progressive censoring data. (For more details about MCMC; see, for example, Robert and Casella (2005) and Recently, Rezaei *et al.* (2010). Recently, Mahmoud *et al.* (2013) have estimated the unknown parameters of generalized Pareto under an adaptive Type-II progressive censoring.

If the failure times of the n items originally on the test are from a Burr Type-XII (Burr-XII) population with cdf $F(x)$ and pdf $f(x)$ as follow:

$$f(x) = \frac{c\beta x^{(c-1)}}{\left(1 + x^c\right)^{\beta+1}} \tag{2.1}$$

And:

$$F(x) = 1 - \frac{1}{\left(1 + x^c\right)^{\beta}}, \quad \beta, c, x > 0. \tag{2.2}$$
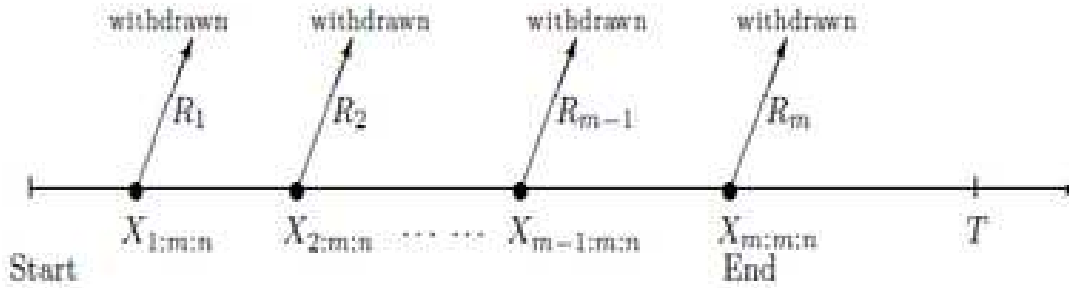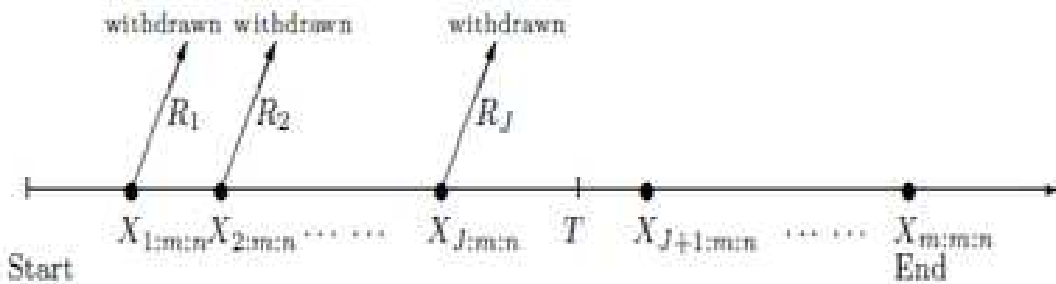
Fig. 1. Progressive Type-II censoring

Fig. 2. Adaptive progressive Type-II censoring

In addition, the hazard rate function and reliability function are:

$$H(x) = \frac{c\beta x^{(c-1)}}{(1+x^c)} \tag{2.3}$$

And:

$$S(x) = \frac{1}{(1+x^c)^\beta}. \tag{2.4}$$

Burr-XII distribution was first introduced in the literature by Burr (1942) and has special importance of using it in practical situations. Rodriguez (1977) showed that the Burr coverage area on a specific plane is occupied by various well known and useful distributions, including the normal, log-normal, gamma, logistic and extreme value type-I distributions. It has been applied in areas of reliability studies and failure time modeling. Other applications include simulation, approximation of distributions and development of non-normal control charts. Many authors have studied inferences on the Burr-XII distribution. Wingo (1993; Moore and Papadopoulos, 2000; Ali Mousa and Jaheen, 2002; Wu and Yu, 2005; Soliman, 2005; Xiuchun *et al.*, 2007; Jaheen and Okasha, 2011; Wu *et al.*, 2014).

Maximum Likelihood Estimation (MLE)

In this section, the confidence interval for the unknown parameters c and β are obtained using MLE technique. Let $x_{1:m:n}$, $x_{2:m:n}, \cdots$, $x_{m:m:n}$ be an adaptive type-II progressive censored order statistics from Burr-XII distribution with censoring scheme $\{R_1, R_2, R_3, \cdots, R_m\}$, then the joint probability density function can written as follows (Ng *et al.*, 2010):

$$f_{1,2,\cdots,m:m:n}\left(x_1,x_2,\cdots,x_m\right) = d_j \left(\prod_{i=1}^{m} f\left(x_{i:m:n}\right)\right)$$
$$\left(\prod_{i=1}^{j}\left(1-F\left(x_{i:m:n}\right)\right)\right)^{R_i} \left(1-F\left(x_{m:m:n}\right)\right)^{n-m-\sum_{i=1}^{J}R_i}, \tag{3.1}$$
$$0 < x_{1:m:n} < x_{2:m:n} < \cdots < x_{m:m:n} < \infty$$

Where:

$$d_j = \prod_{i=1}^{m}\left(n-i+1-\sum_{k=1}^{\max(i-1,J)}R_k\right)$$

The maximum likelihood for Burr-XII distribution can be written from (3.1), as follow:

$$\ell \propto \left(\prod_{i=1}^{m}\frac{c\beta x_{i:m:n}^{(c-1)}}{\left(1+x_{i:m:n}^c\right)^{\beta+1}}\right)\left(\prod_{I=1}^{j}\frac{1}{\left(1+x_{i:m:n}^c\right)^\beta}\right)^{R_i}\left(\frac{1}{\left(1+x_{m:m:n}^c\right)^\beta}\right)^{n-m-\sum_{i=1}^{J}R_i} \tag{3.2}$$

Taking logarithm of ℓ gives:

$$L = \log \ell = m \log c + m \log \beta + (c-1)$$

$$\sum_{i=1}^{m} \log x_{i:m:n} - (\beta+1) \sum_{i=1}^{m} \log(1 + x_{i:m:n}^c)$$

$$-\beta \sum_{i=1}^{j} R_i \log(1 + x_{i:m:n}^c) \tag{3.3}$$

$$-\beta \left(n - m - \sum_{i=1}^{j} R_i \right) \log(1 + x_{m:m:n}^c)$$

The point estimation for c and β can obtain by differentiating (3.3) to c and β; respectively and equating the new equations to zero, yield:

$$\frac{\partial L}{\partial c} = \frac{m}{c} + \sum_{i=1}^{m} \log x_{i:m:n} - (\beta+1) \sum_{i=1}^{m} \frac{x_{i:m:n}^c \log x_{i:m:n}}{1 + x_{i:m:n}^c}$$

$$-\beta \sum_{i=1}^{j} R_i \log \frac{x_{i:m:n}^c \log x_{i:m:n}}{1 + x_{i:m:n}^c} \tag{3.4}$$

$$-\beta \left(n - m - \sum_{i=1}^{j} R_i \right) \frac{x_{m:m:n}^c \log x_{m:m:n}}{1 + x_{m:m:n}^c} = 0$$

$$\frac{\partial L}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^{m} \log(1 + x_{i:m:n}^c) + \sum_{i=1}^{j} R_i \log(1 + x_{i:m:n}^c)$$

$$-\left(n - m - \sum_{i=1}^{j} R_i \right) \log(1 + x_{m:m:n}^c) = 0 \tag{3.5}$$

Solving Equation 3.4 and 3.5 to obtain $\hat{c}$ and $\hat{\beta}$, it's clear that, the analytical solution may be very difficult to find. So, the numerical methods are used to solve these equations. Also, the corresponding MLE of $S(t)$ and $H(t)$ can be written as:

$$\hat{H}(x) = \frac{\hat{c}\hat{\beta}x^{(\hat{c}-1)}}{(1 + x^{\hat{c}})} \tag{3.6}$$

$$\hat{S}(x) = \frac{1}{(1 + x^{\hat{c}})^{\hat{\beta}}} \tag{3.7}$$

Asymptotic Confidence Intervals

In this section, the approximate confidence interval for c, β, $S(t)$ and $H(t)$ are obtained. c, β was approximately bivariate normal with mean (c, β) and variance-covariance matrix I_0^{-1}, Greene (2000; Agresti, 2002).
Where:

$$I_0(c,\beta) = \begin{bmatrix} -\dfrac{\partial^2 L}{\partial c^2} & \dfrac{\partial^2 L}{\partial c \partial \beta} \\ \dfrac{\partial^2 L}{\partial \beta \partial c} & \dfrac{\partial^2 L}{\partial \beta^2} \end{bmatrix} \tag{4.1}$$

The variance-covariance matrix is:

$$I_0^{-1}(c,\beta) = \begin{bmatrix} var\hat{c} & covar(\hat{c},\hat{\beta}) \\ covar(\hat{c},\hat{\beta}) & var\hat{\beta} \end{bmatrix} \tag{4.2}$$

The $(1-\alpha)\%$ approximate confidence interval for c, β, $S(t)$ and $H(t)$; respectively:

$$\left(\hat{c} - Z_{\frac{\alpha}{2}}\sqrt{var\hat{c}}, \hat{c} + Z_{\frac{\alpha}{2}}\sqrt{var\hat{c}} \right),$$

$$\left(\hat{\beta} - Z_{\frac{\alpha}{2}}\sqrt{var\hat{\beta}}, \hat{\beta} + Z_{\frac{\alpha}{2}}\sqrt{var\hat{\beta}} \right),$$

$$\left(\hat{S}(t) - Z_{\frac{\alpha}{2}}\sqrt{var\hat{S}(t)}, \hat{S}(t) + Z_{\frac{\alpha}{2}}\sqrt{var\hat{S}(t)} \right),$$

$$\left(\hat{H}(t) - Z_{\frac{\alpha}{2}}\sqrt{var\hat{H}(t)}, \hat{H}(t) + Z_{\frac{\alpha}{2}}\sqrt{var\hat{H}(t)} \right)$$

where, $var(c)$ and $var(\beta)$ are the variances of the unknown parameters c and β which obtained from inverse fisher information matrix and $Z_{\frac{\alpha}{2}}$ is the percentile of the two tail probability $Z_{\frac{\alpha}{2}}$ calculated from standard normal distribution.

The $\sqrt{var\hat{H}(t)}$ and $\sqrt{var\hat{S}(t)}$ can be calculated from the next equations:

$$\sqrt{var\hat{H}(t)} \simeq \left[G_1^T I_0^{-1} G_1 \right]_{(\hat{c},\hat{\beta})} \tag{4.3}$$

And:

$$\sqrt{var\hat{S}(t)} \simeq \left[G_2^T I_0^{-1} G_2 \right]_{(\hat{c},\hat{\beta})} \tag{4.4}$$

where, $G_1 = \left(\dfrac{\partial H(t)}{\partial c} \quad \dfrac{\partial H(t)}{\partial \beta} \right)$, $G_2 = \left(\dfrac{\partial S(t)}{\partial c} \quad \dfrac{\partial S(t)}{\partial \beta} \right)$ and I_0^{-1} is the variance-covariance matrix defined in (4.2):

Bayes Estimation

This section deals with estimating unknown parameters of Burr-XII distribution using Bayesian procedure. First, let the parameters c and β are independent and have gamma distributions as prior functions in the form; respectively:

$$\pi_1(c) = \frac{b^a c^{a-1} e^{-bc}}{\Gamma(a)}, \; c > 0 \tag{5.1}$$

And:

$$\pi_2(\beta) = \frac{d^k \beta^{k-1} e^{-d\beta}}{\Gamma(k)}, \; \beta > 0 \tag{5.2}$$

The joint prior distribution of c and β can be given as:

$$g(c,\beta) = \ell(c,\beta \mid \underline{x}) \times \pi_1(c) \times \pi_2(\beta) \qquad (5.3)$$

The joint posterior density function can be written as:

$$\pi(c,\beta \mid \underline{x}) = \frac{\ell(c,\beta \mid \underline{x}) \times \pi_1(c) \times \pi_2(\beta)}{\int_0^\infty \int_0^\infty \ell(\underline{x} \mid c,\beta) \times \pi_1(c) \times \pi_2(\beta) dc d\beta} \qquad (5.4)$$

The Bayes estimator of any function under squared error loss function is the posterior mean and giving by:

$$\hat{g}(c,\beta \mid \underline{x}) = E\big(g(c,\beta \mid \underline{x})\big)$$

$$= \frac{\int_0^\infty \int_0^\infty g(c,\beta)\ell\big((c,\beta \mid \underline{x})\big) \times \pi_1(c) \times \pi_2(\beta) dc d\beta}{\int_0^\infty \int_0^\infty \ell(c,\beta \mid \underline{x}) \times \pi_1(c) \times \pi_2(\beta) dc d\beta} \qquad (5.5)$$

In general, Equation 5.5 can't reduce to closed form. The exact solution may be very hard. In this case, the MCMC method to generate sample of (5.4) maybe the best procedure to compute Bayes estimator of $g(c, \beta)$.

Next section shows the MCMC approach including Metropolis-Hastings within Gibbs Sampling technic.

Bayesian Estimation Using MCMC Approach

The Metropolis-Hastings algorithm is a very general MCMC method developed by Metropolis *et al*. (1953) and extended by Hastings (1970). It's clear that, the solution of (5.5) is very hard to calculate, so, the MCMC maybe suitable method to estimate $\hat{c}$ and $\hat{\beta}$ by Bayesian approach. In order to use the method of MCMC for estimating the parameters of the Burr-XII distribution, let us consider independent priors (5.1 and 5.2), respectively, for the parameters c and β. The joint posterior density function can be obtained up to proportionality by multiplying the likelihood with the prior and this can be written as:

$$\pi(c,\beta \mid \underline{x}) \propto \beta^{m+k-1} c^{m+a-1} e^{-bc} e^{-d\beta} \left(\prod_{i=1}^m \frac{x_{i:m:n}^{(c-1)}}{\left(1+x_{i:m:n}^c\right)^{\beta+1}} \right)$$
$$\times \left(\prod_{i=1}^j \frac{1}{\left(1+x_{i:m:n}^c\right)^\beta} \right)^{R_i} \left(\frac{1}{\left(1+x_{m:m:n}^c\right)^\beta} \right)^{n-m-\sum_{i=1}^j R_i} \qquad (6.1)$$

From (6.1), the posterior density function of c given β and β given c are, respectively:

$$g_1(c \mid \beta, \underline{x}) \propto c^{m+a-1} e^{-bc} \left(\prod_{i=1}^m \frac{x_{i:m:n}^{(c-1)}}{\left(1+x_{i:m:n}^c\right)^{\beta+1}} \right)$$
$$\times \left(\prod_{i=1}^j \frac{1}{\left(1+x_{i:m:n}^c\right)^\beta} \right)^{R_i} \times \left(\frac{1}{\left(1+x_{m:m:n}^c\right)^\beta} \right)^{n-m-\sum_{i=1}^j R_i} \qquad (6.2)$$

And:

$$g_2(\beta \mid c, \underline{x}) \propto \beta^{m+k-1} e^{-d\beta} \left(\prod_{i=1}^m \frac{x_{i:m:n}^{(c-1)}}{\left(1+x_{i:m:n}^c\right)^{\beta+1}} \right)$$
$$\times \left(\prod_{i=1}^j \frac{1}{\left(1+x_{i:m:n}^c\right)^\beta} \right)^{R_i} \left(\frac{1}{\left(1+x_{m:m:n}^c\right)^\beta} \right)^{n-m-\sum_{i=1}^j R_i} \qquad (6.3)$$

It's clear that the posterior density function of c and β given in (6.2 and 6.3) cannot be reduced analytically to well known distributions. The Metropolis-Hastings method with normal proposal distribution will be used to generate c and β from the posterior density functions and in turn obtain the Bayes estimates and the corresponding credible intervals can be written as:

- Start with $\beta^{(0)}$
- Set $v = 1$
- Using Metropolis-Hastings (Metropolis *et al*., 1953), generate c^v from g_1 with the normal $N(c^v, \sigma^2)$. Where σ^2 is variances-covariance matrix
- Also, generate β^v from g_2 with the normal $N(\beta^{v-1}, \sigma^2)$
- Compute c^v and β^v
- Set $v = v + 1$
- Repeat 3-6 steps N times
- Rearrange the values c^i and β^i, $i = M + 1, M + 2, \cdots, N$
- Obtain the Bayes estimates of c and β with respect to the squared error loss function as:

  $$E(c \mid data) = \frac{1}{N-M} \sum_{i=M+1}^N c_i$$
  and
  $$E(\beta \mid data) = \frac{1}{N-M} \sum_{i=M+1}^N \beta_i, \text{ where M is burn-in}$$

- $(1-\alpha)\%$ credible intervals of c and β can be computed as:

  $$\left(c_{\left\{\frac{\alpha}{2}(N-M)\right\}}, c_{\left\{\frac{1-\alpha}{2}(N-M)\right\}} \right) and \left[\beta_{\left\{\frac{\alpha}{2}(N-M)\right\}}, \beta_{\left\{\frac{1-\alpha}{2}(N-M)\right\}} \right]$$

Numerical Results

This section deals with obtaining some numerical results. Confidence interval for c and β using maximum likelihood and Bayes include MCMC technique are calculated. In order to compare the different estimators

of the parameters, 11000 an adaptive Type-II progressive samples from the Burr-XII distribution with different censoring schemes R are simulated. The samples were simulated by using the algorithm described in Ng *et al*. (2010). Compare the performances of ML and Bayesian estimates in terms of Mean Squared Errors (MSEs). To find the Bayesian MCMC estimates, the non-informative gamma priors is used, call it prior 0. In addition, an informative gamma priors for the two parameters call it prior1 is used.

Table 1. Censoring schemes with different values for *n* and *m*

Censoring scheme	n	m	R
CS_1	50	30	$R_1 = \{0^8, 2, 0^6, 8, 3, 0^3, 5, 0^8, 2\}$
CS_2	50	30	$R_2 = \{0^4, 1, 0^6, 3, 2, 0^2, 8, 3, 0^3, 3, 0^9\}$
CS_3	60	30	$R_3 = \{0^3, 5, 0, 8, 0^2, 2, 0^2, 7, 0^3, 1, 3, 0^3, 1, 0^5, 1, 0^2, 2\}$
CS_4	60	30	$R_4 = \{0^2, 5, 0, 4, 0^4, 3, 0^3, 2, 0, 3, 0^3, 5, 0^3, 3, 0^2, 5, 0^3\}$
CS_5	60	40	$R_5 = \{0^8, 2, 0^3, 6, 0^2, 0^5, 3, 0^2, 2, 0^4, 1^2, 0^3, 2, 0^3, 2\}$
CS_6	60	40	$R_6 = \{0^4, 3, 0^5, 2, 0^4, 2, 3, 0^3, 5, 0^8, 2, 1, 0^8, 2\}$
CS_7	60	50	$R_7 = \{0^8, 2, 0^6, 1, 3, 0^3, 1, 0^8, 2, 1, 0^{19}\}$
CS_8	60	50	$R_8 = \{0^8, 2, 0^6, 3, 0^{19}, 0^{10}, 2, 0^3\}$
CS_9	70	50	$R_9 = \{0^8, 2, 0^3, 3, 0^3, 3, 0^3, 7, 0^8, 2, 3, 0^{19}\}$
CS_{10}	70	50	$R_{10} = \{0^8, 2, 0^6, 8, 3, 0^3, 5, 0^8, 2, 0^{20}\}$
CS_{11}	70	60	$R_{11} = \{0^8, 2, 0^6, 1, 3, 0^{12}, 2, 0^8, 2, 0^{21}\}$
CS_{12}	70	60	$R_{12} = \{0^{15}, 2, 0^4, 5, 0^{18}, 3, 0^{20}\}$

Remark: 0^r means that 0 repeated *r* times.

Table 2. Mean, Interval Length (IL) for *c* and *β* using Maximum Likelihood method (ML)

		Censoring scheme											
		CS_1	CS_2	CS_3	CS_4	CS_5	CS_6	CS_7	CS_8	CS_9	CS_{10}	CS_{11}	CS_{12}
$\hat{c}$	mean	3.1068	3.2235	3.131	3.2061	3.0988	3.1953	3.0756	3.0772	3.1024	3.0443	3.0992	3.1028
	MSE	0.2723	0.4136	0.3537	0.3344	0.2481	0.2521	0.2479	0.2161	0.2574	0.1901	0.1812	0.2476
	IL	2.1949	2.2742	2.0641	2.1351	1.9282	2.0240	1.8794	1.8769	1.8010	1.7354	1.7753	1.7746
	cov	0.9800	0.9700	0.9200	0.9800	0.9700	0.9900	0.9700	0.9500	0.9600	0.9500	0.9800	0.9700
$\hat{\beta}$	mean	0.4995	0.5167	0.5330	0.5097	0.5111	0.4957	0.5057	0.5122	0.4945	0.5136	0.4903	0.4949
	MSE	0.0109	0.0156	0.0117	0.0089	0.0084	0.0094	0.0084	0.0097	0.0075	0.0082	0.0078	0.0079
	IL	0.4092	0.4191	0.4097	0.3968	0.3733	0.3666	0.3590	0.3620	0.3350	0.3433	0.3247	0.3267
	cov	0.9600	0.9100	0.9600	0.9600	0.9800	0.9400	0.9700	0.9500	0.9100	0.9500	0.9200	0.9200
$\hat{S}(t)$	mean	0.9684	0.9693	0.9664	0.9705	0.9680	0.9714	0.9677	0.9672	0.9688	0.9667	0.9696	0.9688
	MSE	0.0171	0.0187	0.0184	0.0158	0.0166	0.0148	0.0148	0.0171	0.0161	0.0160	0.0149	0.0158
	IL	0.0691	0.0668	0.0663	0.0616	0.0635	0.0592	0.0639	0.0641	0.0580	0.0604	0.0571	0.0578
	cov	0.8700	0.8800	0.8800	0.8800	0.9300	0.8800	0.9100	0.9200	0.8900	0.9500	0.8600	0.8800
$\hat{H}(t)$	mean	0.2226	0.2194	0.2359	0.2147	0.2266	0.2100	0.2282	0.2314	0.2204	0.2336	0.2180	0.2212
	MSE	0.0881	0.0972	0.0933	0.0834	0.0831	0.0822	0.0794	0.0872	0.0810	0.0800	0.0780	0.0826
	IL	0.0269	0.0273	0.0352	0.0260	0.0309	0.0249	0.0310	0.0333	0.0321	0.0364	0.0305	0.0327
	cov	0.9200	0.9000	0.9000	0.9100	0.9200	0.9200	0.9300	0.8900	0.9000	0.9600	0.9000	0.8900

Table 3. Mean, Interval Length (IL) for *c* and *β* using Bayesian method (mcmc1)

		Censoring scheme											
		CS_1	CS_2	CS_3	CS_4	CS_5	CS_6	CS_7	CS_8	CS_9	CS_{10}	CS_{11}	CS_{12}
$\hat{c}$	mean	2.9655	3.0755	3.0039	3.052	2.9993	3.0529	2.9543	2.9402	2.9988	2.9251	2.9946	2.9800
	MSE	0.2787	0.5322	0.3832	0.3814	0.3465	0.3294	0.2921	0.2281	0.2662	0.2274	0.1954	0.3220
	IL	1.1237	1.0819	1.0364	1.0518	0.9601	1.0037	0.9694	0.9610	0.9318	0.9037	0.8870	0.8507
	cov	0.6500	0.5400	0.5200	0.6200	0.5500	0.5700	0.6200	0.6400	0.6300	0.6400	0.6700	0.5800
$\hat{\beta}$	mean	0.5189	0.5371	0.5465	0.5261	0.5274	0.5163	0.5263	0.5342	0.5103	0.5321	0.5077	0.5161
	MSE	0.0117	0.0177	0.0128	0.0089	0.0100	0.0104	0.0100	0.0110	0.0087	0.0099	0.0082	0.0094
	IL	0.3842	0.3958	0.3932	0.3810	0.3401	0.3360	0.3203	0.3251	0.3018	0.3141	0.2856	0.2855
	cov	0.9200	0.8600	0.9600	0.9900	0.9400	0.9000	0.8900	0.8700	0.8400	0.8800	0.8600	0.8600
$\hat{S}(t)$	mean	0.9628	0.9617	0.9607	0.9644	0.9624	0.9650	0.9619	0.9614	0.9642	0.9613	0.9653	0.9626
	MSE	0.0201	0.0260	0.0240	0.0200	0.0222	0.0199	0.0205	0.0212	0.021	0.0201	0.0178	0.0217
	IL	0.0774	0.0766	0.0729	0.0695	0.0699	0.0673	0.0710	0.0715	0.0632	0.0667	0.0625	0.0646
	cov	0.9300	0.8500	0.9300	0.9100	0.9300	0.9100	0.9200	0.9300	0.9000	0.9100	0.9100	0.8800
$\hat{H}(t)$	mean	0.2489	0.2512	0.2589	0.2413	0.2494	0.2388	0.2540	0.2585	0.2408	0.2584	0.2386	0.2481
	MSE	0.0972	0.1197	0.1102	0.0937	0.1014	0.0977	0.0974	0.1004	0.0943	0.0943	0.0858	0.1008
	IL	0.3748	0.3703	0.3495	0.3426	0.3397	0.3363	0.3443	0.3491	0.3122	0.3226	0.3143	0.3137
	cov	0.9300	0.8600	0.9400	0.9000	0.9000	0.9100	0.9200	0.9000	0.8700	0.9200	0.9300	0.8800

Table 4. Mean, Interval Length (IL) for c and β using non-informated Bayesian method (mcmc0)

		Censoring scheme											
		CS_1	CS_2	CS_3	CS_4	CS_5	CS_6	CS_7	CS_8	CS_9	CS_{10}	CS_{11}	CS_{12}
$\hat{c}$	mean	3.0734	3.1983	3.1837	3.1383	3.0805	3.1747	3.0991	3.0468	3.0875	3.0194	3.0635	3.0571
	MSE	0.3650	0.4544	0.5333	0.3339	0.3265	0.3528	0.3594	0.3145	0.3628	0.2654	0.2366	0.3440
	IL	1.1341	1.1668	1.0806	1.1073	1.0262	1.0003	0.9970	0.9777	0.9260	0.9119	0.8714	0.8596
	cov	0.6300	0.6400	0.6000	0.6100	0.5700	0.6200	0.6100	0.5600	0.4800	0.6600	0.6000	0.5600
$\hat{\beta}$	mean	0.5045	0.5204	0.5289	0.5142	0.5149	0.5011	0.5082	0.521	0.4993	0.5191	0.4976	0.5047
	MSE	0.0110	0.0163	0.0121	0.0087	0.0089	0.0110	0.0105	0.0123	0.0085	0.0085	0.0082	0.0093
	IL	0.3794	0.3896	0.3888	0.3771	0.3412	0.3309	0.3135	0.3195	0.2971	0.3114	0.2781	0.2817
	cov	0.9100	0.8800	0.9500	0.9700	0.9200	0.8700	0.8700	0.8200	0.8600	0.9100	0.8400	0.8700
$\hat{S}(t)$	mean	0.9659	0.9678	0.9664	0.9681	0.9661	0.9690	0.9664	0.9639	0.9666	0.9649	0.9674	0.9655
	MSE	0.0207	0.0200	0.0200	0.0178	0.0192	0.0188	0.0191	0.0222	0.0192	0.0178	0.0169	0.0193
	IL	0.0718	0.0689	0.0655	0.0645	0.0653	0.0616	0.0647	0.0672	0.0599	0.0624	0.0596	0.0611
	cov	0.9000	0.8400	0.8100	0.8700	0.8900	0.8700	0.8600	0.8500	0.8500	0.9300	0.8700	0.8800
$\hat{H}(t)$	mean	0.2308	0.2263	0.2329	0.2244	0.2331	0.2188	0.2314	0.2436	0.2281	0.2411	0.2274	0.2346
	MSE	0.0984	0.1028	0.1029	0.0872	0.0927	0.0965	0.0965	0.1068	0.0934	0.0865	0.0861	0.0955
	IL	0.0326	0.0308	0.0370	0.0306	0.0357	0.0312	0.0359	0.0423	0.0376	0.0405	0.0355	0.0403
	cov	0.9000	0.8800	0.8700	0.9100	0.9000	0.8600	0.8700	0.8500	0.8900	0.9500	0.8800	0.9000

Table 5. Maximum likelihood and non-informative Bayesian including Markov chain Mont Carlo simulation (mcmc0) For real data set

ML	Mean	Lower	Upper
c	1.2739	0.3836	2.1642
β	0.2484	0.0370	0.4599
S	0.7369	0.5883	0.8854
H	0.1120	0.0350	0.1889
mcmc0			
c	1.3163	0.5882	2.3965
β	0.2581	0.0938	0.5282
S	0.7242	0.5647	0.8837
H	0.1212	0.0392	0.2032

The Bayesian estimates and probability intervals based on 11000 MCMC samples and discard the first 1000 values as burn-in is computed. All results are given in Tables are obtained at $c = 3$; $\beta = 0.5$; $T = 0.5$; $t = 0.4$; $\alpha = 0.05$; $a = 0.5$; $b = 1$; $\gamma = 0.8$; $\xi = 1$; $nmcmc = 11000$; $nburn = 1000$; $a1 = 0$; $b1 = 0$; $\xi_1 = 0$; $\gamma_1 = 0$ and different censoring schemes (Table 1).
where, 0^r means that 0 repeated r times.

Remarks

From Table 2-4, observe that:

- When n and m increase, the MSE specially in mcmc1 and ML decrease
- In mcmc1 and mcmc0, the coverage probability is small than ML for parameter c
- Based on the interval length and the mean squared error. The mcmc1 is the best method for estimating all parameters except the parameter c

Illustrative Example

The data set was discussed here are presented by (Zimmer *et al*., 1998; Lio *et al*., 2010) for the Burr-XII reliability analysis. Lio *et al*. (2010) studied the validity of the model for both data sets and they showed that Burr-XII distribution fits quite well for both the data sets. These data sets are 0.19, 0.78, 0.96, 0.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71 and 72.89. It is clear that $n = 19$. Suppose that, $m = 10$; $\alpha = 0.05$ (significant level); $t = 2$; $nmcmc = 11000$; $nburn = 1000$, $R = 1, 0, 0, 2, 0, 3, 0, 0, 1, 2$. If $T = 7.5$ then $J = 8$ and $R = 1, 0, 0, 2, 0, 3, 0, 0, 0, 3$ then $x_i(i = 1, 2,\cdots,m) = 0.19, 0.31, 0.96, 2.78, 3.16, 4.15, 4.67, 6.5, 8.27, 31.75$. Because no prior information about unknown parameters. The results are calculated here using mathematica 8 and posted in Table 5 using maximum likelihood and non-informative Bayesian including Markov chain Mont Carlo simulation (mcmc0).

Conclusion

In this study, the Maximum Likelihood Estimation (MLE) and Bayes estimation are exploited to make interval estimation for the unknown parameters of Burr-XII distribution. Bayes estimator does not exist in an explicit form for the parameters, Markov Chain Monte Carlo (MCMC) method is used to generate samples from the posterior distribution. Numerical example using real data set is presented to illustrate the methods of inference developed here.

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Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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