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# On Hyper d-Algebras

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**Abstract:** After the introduction of hyper algebra different hyper algebraic structures like hyper BCK-algebra, hyper BE-algebra, hyper UP-algebra and hyper KU-algebras were introduced. In this research paper, we introduce the concept of Hyper d-Algebra which is a generalization of a d algebra, and investigate some related properties. Moreover, we discuss Hyper d-Algebra, the sub-algebra of Hyper d-Algebra, the ideal of Hyper d-Algebra and we discuss the Cartesian product of Hyper d-Algebra.

**Keywords:** d-Algebra, Hyper d-Algebra, Hyper Sub Algebra of Hyper d-Algebra, Hyper Ideal

## Introduction

Hyperstructures have many applications in several sectors of both pure and applied parts of mathematics by Jun (1999; 2001). A good reference for the theory of hyper-structures and its applications to Mathematics and Computer Science can be found in Torkzadeh and Zahedi (2006); Dejen (2020). Jun and Zahedi (2000) applied the hyperstructures to BCK-algebras and the concept of hyper BCK-algebras which is a generalization of BCK-algebras and investigated some related properties.

Akefe Radfar, Akbar Rezaei, and Arsham Borumand Saeid (Radfar *et al*., 2014) discussed hyper BE algebra and some related concepts (Sh. Ghorbani and Eslami, 2008) applied the hyperstructures to KU-algebras. Neggers *et al*. (1999); Mostafa *et al*. (2017) discussed the notion of d-algebras as a generalization of BCK-algebra and investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs. Gerima (2022) introduced the concepts of fuzzy dot d-sub algebras and fuzzy dot d-ideals of a d-algebra. The product of fuzzy dot d-ideals and strong fuzzy relation and the corresponding strong fuzzy dot d-ideal was discussed. Neggers *et al*. (1999) discussed the concepts of a fuzzy dot hyper K-subalgebra, a fuzzy dot hyper K-ideal with some other properties.

In this study, we introduce the concepts of Hyper d-Algebra and some related properties of Hyper d-Algebra.

## Materials and Methods

In this section some basic definitions and basic examples that help to clarify the new concepts are included. We used the methods introduced in d-algebra and hyper BCK-algebras.

Definition 2.1 (Neggers *et al*., 1999) A d-algebra is a non-empty set $X$ with a constant 0 and a binary operation $*$ satisfying the following axioms:

$$x * x = 0 \quad (1)$$

$$0 * x = 0 \quad (2)$$

$$x * y = 0 \text{ and } y * x = 0 \text{ Imply that } x = y, \text{ for } x, y \text{ in } X \quad (3)$$

Example 2.2. Let $X = \{0, a, b, c\}$ be *a* set. Then the operation $*$ on $X$ is defined by Table 1.

Clearly $(X, *, 0)$ is a d-algebra.

Definition 2.3 (Mostafa *et al*., 2017) Let $(X, *, 0)$ be a d-algebra $\emptyset \neq I \subseteq X$ then:

1. $I$ is called a d-sub algebra of $X$ if $x{\star}y \in I$ whenever $x \in I$ and $y \in I$
2. $I$ is called a d-ideal of $X$ satisfies

$$D_0.\ 0 \in I \quad (1)$$

$$D_1.\ x * y \in 1 \text{ and } y \in 1 \text{ imply } x \in 1 \quad (2)$$

$$D_2.\ x \in 1 \text{ and } y \in X \text{ imply } x * y \in 1 \quad (3)$$

Definition 2.4 (Jun 1999) A hyper BCK-algebra with a binary operation o and a constant 0 satisfying the following axioms:

$$(x \circ z) \circ (y \circ z) << (x \circ y) \quad (1)$$

$$(xoy)oz = (xoz)oy \quad (2)$$

$x << y$ and $y << x$ implies that:

$$x = y \ \ for\ x, y \in H \quad (3)$$

**Table 1:** Hyper d-Algebra

| * | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | a | c |
| b | b | b | 0 | b |
| c | c | a | c | 0 |

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## Results

### *Hyper d-Algebras*

Definition 3.1.1. Let $H$ be a non-empty set and $o: H \times H \rightarrow P(H) \backslash \{\phi\}$ be a hyperoperation. Then $(H, o, 0)$ is called a Hyper d-Algebra if it satisfies the following axioms:

$$x << x \quad (1)$$

$$0 << x \quad (2)$$

If $x << y$ *and* $y << x$, then $x = y$ for all:

$$x, y \in H \quad (3)$$

Remark 3.1.2. Let $H$ be a Hyper d-Algebra. Then the following properties hold.

$x \circ y$ can be written as:

$$x \circ \{y\}, \{x\} \circ \{y\}, \{x\} \circ y \quad (1)$$

If $\varphi \neq A, B \subseteq H$, then $A \circ B = \bigcup_{(a,b) \in A \times B} (a \circ b)$, then $A << B$ means for all $a \in A$ there exists $b \in B$ such that:

$$a << b \quad (2)$$

$$P(H) \text{ denote the set of all subsets of } H \quad (3)$$

For $x, y \in H$ $x << y$ means $0 \in x \circ y$.

Example 3.1.3. Let $H = \{0, a, b\}$ be a set. Then the hyperoperation "$o$" on $H$ is defined by Table 2.

By simple manipulation $(H, o, 0)$ is a hyper d-Algebra.

Proposition 3.1.4. Let $H$ be a Hyper d-Algebra. Then for all $x \in H$ and $A, B \subseteq H$. Then each of the following conditions holds:

$$0o0 = \{0\} \quad (1)$$

$$A << \{0\} \text{ implies } A = \{0\} \quad (2)$$

$$A \subseteq B \text{ implies } A << B \quad (3)$$

$$\{0\} \subseteq xo(x\, ox) \quad (4)$$

Proof 1. Let $H$ be a Hyper d-Algebra. Then $x << x$, for all $x \in H$.

We want to show that $0o0 = \{0\}$ Since:

$$x << x \Rightarrow 0 \in x \circ x \quad (1)$$

If we put $x = 0$ in the form $x << x$ we get $0 \circ 0 = \{0\}$ *Thus*, $0o0 = \{0\}$.

Suppose that $H$ is a Hyper d-Algebra, for all $x, y \in H$ *and* $A \subseteq H$.

**Table 2:** Hyper d-Algebra

| o | 0 | a | b |
|---|---|---|---|
| 0 | {0} | {0, a} | {0, b} |
| a | {a} | {0, a} | {a, b} |
| b | {b} | {0, b} | {0, a, b} |

$$\textit{We need to show that } A = \{0\}$$

Let for all $x \in A$ there exist $y = 0 \in B$ such that:

$$x << y \quad \textit{imply} \quad x << y = 0 \text{ we get } x = 0$$

Implies $x \in A$. Hence $A = \{0\}$.

Suppose that $H$ is a hyper d-algebra, for all $x \in H$ and $A, B \subseteq H$ *and* $A \subseteq B$:

$$\textit{We need to show that } A << B$$

For all $x \in A$. Then there exists $x \in B$ such that $x \ll x$. Thus $A << B$.

But the converse is not true.

Suppose that $H$ is a hyper d- algebra, for all $x \in H$. We need to show that $\{0\} \subseteq (xox)ox$:

$$0 \in xox \quad (1)$$

$$\{0\} = 0ox(xox)ox \quad (2)$$

$$\{0\} \subseteq (xox)ox \quad (3)$$

Thus, $0 \in (xox)ox$.

### *Sub-Algebra of Hyper d-Algebras*

Definition 4.1.1. Let $(H, o, 0)$ be a Hyper d-Algebra and $S$ be a non-empty subset of $H$ containing 0 with respect to the hyper operation "$o$" which implies that xoy $\ll S$, for all $x, y \in S$. Then $S$ is called a hyper sub-algebra of $H$.

Example 4.1.2. Let $H = \{0,1,2,3\}$ be a set. Then the hyperoperation on $H$ is defined by Table 3.

Clearly $(H, o, 0)$ is a Hyper d-Algebra.

Let $S = \{0, 1, 2\}$ be a subset of a Hyper d-Algebra $(H, o, 0)$.

Since:

$$1o2 = \{1,2\} << S, 1 \in S \textit{ and } 2 \in S \quad (1)$$

$$2o2 = \{0,2\} << S = \{0,1,2\}, 0 \in S \textit{ and } 2 \in S \quad (2)$$

$$1o1 = \{1\} << S = \{0,1,2\}, 1 \in S \quad (3)$$

Thus, $S$ is a sub-algebra of a Hyper d-Algebra.

Proposition 4.1.3. Let $S$ be a non-empty subset of a Hyper d-Algebra $(H, o, 0)$ and $xox \ll S$, for all $x \in S$. Then $0 \in S$.

Proof. Let $S$ be a non-empty subset of a Hyper d-Algebra $(H, o, 0)$ and $xox \ll S$, for all $x \in S$. Let $x \in S$ be arbitrary. Then $xox \ll S \Rightarrow 0 \in xox \subseteq S$. Therefore, $0 \in S$.

**Table 3:** Hyper d-subalgebra

| o | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | {0} | {0,1} | {0,2} | {0,3} |
| 1 | {1} | {0,1} | {1,2} | {1,3} |
| 2 | {2} | {0,2} | {0,2} | {0,2,3} |
| 3 | {3} | {1,3} | {2,3} | {0,1,2,3} |

Theorem 4.4. Let $(H, \mathrm{o}, 0)$ be a Hyper d-Algebra and $S = \{x \in H \mid 0\ o\ x \ll \{0\}\}$. Then $S$ is a Hyper d-subalgebra of $H$.

Proof. Let $(H, o, 0)$ be a hyper d-algebra and $S = \{x \in H \mid 0\ o\ x \ll \{0\}\}$ and let $x, y \in$ S. Then $x= 0\ o\ x \ll \{0\}$ and $y = 0\ o\ y \ll \{0\}$.

Now $xoy = (0ox)o(0oy) \ll \{0\}o\{0\} = \{0\}$, Imply that $xoy \ll \{0\} \Rightarrow xoy \ll S$. Thus, $S$ is a hyper d-subalgebra of $H$.

## *Hyper d-Ideals of Hyper d-Algebras*

Definition 5.1.1. Let $I$ be a non-empty subset of a Hyper d-Algebra $H$. Then $I$ is called a Hyper d-Ideals of $H$ if it satisfies the following axioms:

$$I_0.\ 0 \in I \quad (1)$$

$$I_1.\ x \circ y << I \ \ and \ \ x \in I \ \ imply \ \ y \in I \quad (2)$$

$$I_2.\ x \in I \ \ and \ \ y \in H \ \ imply \ \ x \circ y << I, for\ all\ x,y \in H \quad (3)$$

Example 5.1.2. Let $H = \{0, a, b, c, d\}$ be a set. Then the hyperoperation "$o$" on $H$ is defined by Table 4.

With simple calculation $(H, o, 0)$ is a Hyper d-Algebra.

Let $I= \{0, a, b, c\}$ be a subset of $H$. Then $I$ is a Hyper d-Ideal of $H$. Since:

$$I_0.\ 0 \in I \quad (1)$$

$$I_1. a \circ b << I \ \ and \ \ a \in I\ imply b \in I \quad (2)$$

$$I_2.\ c \in I \ \ and \ \ d \in H\ imply c \circ d << I \quad (3)$$

Thus, $I$ is a Hyper-d-Ideal of $H$.

Proposition 5.1.3. Let $I$ be a Hyper d-Ideal of $H$ and let $A$ be a subset of a Hyper d-Algebra $H$ such that $A \ll I$. Then $A \subseteq I$.

Proof. Let $I$ be a Hyper d-Ideal of $H$ and let $A$ be a subset of $H$ such that $A \ll I$. Then for all $a \in A$ there exist $x \in I$ such that $a \ll x \Rightarrow 0 \in aox \ll I$. $x \in I$ imply $a \in I$, (Since $I$ is a Hyper d-Ideal of $H$). Thus, $A \subseteq I$.

Example 5.1.4. Let $H = \{0, a, b, c\}$ be a set. Then the hyperoperation "$o$" on $H$ is defined by Table 5.

Clearly $(H, o, 0)$ is a Hyper d-Algebra.

Let $I = \{0, a, b\}$ be a non-empty subset of a Hyper d-Algebra $H$. Then:

$$I_0.\ 0 \in I \quad (1)$$

$$I_1. a \circ b = \{a,b\} << I = \{0,a,b\}\ and\ a \in I \ \ \ imply \ \ \ b \in I \quad (2)$$

$$I_2.\ \ a \in I \ \ and \ \ c \in H\ imply\ a \circ c << I \quad (3)$$

Thus, $I$ is a Hyper-d-Ideals of a Hyper-d-Algebra.

Lemma 5.1.5. If $I$ is a Hyper d-Ideal of a Hyper d-Algebra $H$, then $0 \in I$.

**Table 4:** Hyper d-IDEAL of H

| 0 | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | {0} | {0, 0, a} | {0, 0, b} | {0, 0, c} | {0, 0, 0, 0, c} |
| a | {a} | {0, a, b} | {0, a, b} | {0, b, c} | {0, 0, 0, b, c} |
| b | {b} | {0, 0, b} | {0, 0, b} | {0, b, c} | {0, 0,0, 0, a} |
| c | {c} | {0, a, c} | {0, c, a} | {0, 0, c} | {0, 0, 0, c, a} |
| d | {d} | {0, a, b} | {0, b, c} | {0, a, b} | {0, a, b, c, d} |

**Table 5:** Hyper d-Ideal

| 0 | 0 | a | b | c |
|---|---|---|---|---|
| 0 | {0} | {0, 0, a} | {0, b} | {0, b} |
| a | {a} | {0, 0, a,} | {a, b} | {0, b} |
| b | {b} | {0, 0, b} | {0, b} | {0, b} |
| c | {c} | {0, a, b} | {0, a} | {0, 0} |

Proof. Assume $I$ am a Hyper d-Ideal of a Hyper d-Algebra $H$.

Since $I \neq \emptyset$, for all $x \in I$, $x \ll\ x$, we have $0 \in xox \subseteq I$. Thus, $0 \in I$.

Proposition 5.1.6. Let $I$ be a Hyper d-Ideal of a Hyper d-Algebra $H$. If $y \ll x$ and $x \in I$, then $y \in I$.

Proof. Let $I$ be a Hyper d-Ideal of a Hyper d-Algebra $H$ such that $y \ll x$ and $x \in I$.

Since $y \ll x \Rightarrow 0 \in y\ o\ x \subseteq \mathrm{I} \Rightarrow\ 0 \in I$. (proposition 5.1.3.)

Consequently, $yox \ll\ I$ and $x \in I$ imply $y \in I$. Therefore, $y \in I$.

Definition 5.1.7. Let $H$ be $a$ Hyper d-Algebra. Then $a$ hyper d-ideal $I$ of $H$ is called a Hyper $d^{\#}$-ideal of $H$. $xoz\ \ll I$ whenever $xoy \ll I$ and $yoz \ll I$, for arbitrary $x$, $y$, $z \in H$.

Example 5.1.8. Let $H= \{0, a, b, c\}$ be a set. Then the hyperoperation "$o$" on $H$ is defined by Table 6.

Clearly $(H, o, 0)$ is a hyper d- Algebra. Let $I= \{0, a, b\}$ be a subset of a hyper d-Algebra. Then:

$$a \circ c = \{0,b,a\} << I, whenever\ a \circ a = \{0,a\} << I \\ and\ a \circ c = \{0,b,a\} << I \quad (1)$$

$$a \circ 0 = \{a\} << I, whenever\ a \circ b = \{a,b\} << I \\ and\ b \circ c = \{b\} << I \quad (2)$$

$$a \circ c = \{0,b,a\} << I, whenever\ a \circ a = \{0,a\} << I \\ and\ \ a \circ c = \}0,b,a\} << I \quad (3)$$

$$a \circ b = \{a,b\} << I, whenever\ a \circ c = \{0,b,a\} << I \\ and\ c \circ b = \{0,a\} << I \quad (4)$$

$Thus, I$ is a hyper $d^{\#} -$ ideal of $H$.

Definition 5.1.9. Let $I$ be a Hyper $d^{\#}$-Ideal of a Hyper d-Algebra $H$ satisfies $xoy \ll I$ and $yox \ll I$ imply $(xoz)o(yoz) \ll\ I$ and $(zox)o(zoy) \ll$ I, for all $x, y \in H$. Then $I$ is called a Hyper $d^{*}$-Ideal of $H$.

Example 5.1.10. Let $H$ = {0, 1, 2, 3, 4} be a set. Then the hyperoperation "$o$" on $H$ is defined by Table 7.

Clearly $(H, o, 0)$ is a Hyper d-Algebra.

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**Table 6:** Hyper d*-ideal

| o | 0 | a | b | c |
|---|---|---|---|---|
| 0 | {0} | {0, a} | {0, b} | {0, 0, a} |
| a | {a} | {0, a} | {a, b} | {0, b, a} |
| b | {b} | {0, b} | {0, b} | {0, 0, b} |
| c | {c} | {0, a} | {0, a} | {0, a, c} |

**Table 7:** Hyper d*-ideal

| o | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | {0} | {0, 0, 1} | {0, 2} | {0, 2} | {0, 0, 0} |
| 1 | {1} | {0, 0, 1} | {1, 2} | {0, 0} | {0, 0, 1} |
| 2 | {2} | {0, 1, 2} | {0, 2} | {1, 2} | {0, 0, 2} |
| 3 | {3} | {0, 1, 2} | {0, 1} | {0, 1} | {0, 0, 2} |
| 4 | {4} | {0, 0, 2} | {0, 1} | {0, 1} | {0, 1, 2} |

Let $I$= {0,1,2,3} be a subset of a Hyper d- Algebra.

Since, $0o1= \{0,1\} \ll I$ and $1o0= \{0,1\} \ll I$ imply $(0o2)o(1o2) = \{0,1,2\} \ll I = \{0,1,2,3\}$ and $(2o0)o(2o1) \ll I = \{0,1,2\} \ll I = \{0,1,2,3\}$ and $3o4 = \{0,2\} \ll I$ and $4o3= \{0,1\} \ll I$ imply $(3o3)o(4o3) = \{0,1\} \ll I = \{0,1,2,3\}$ and $(3o3)o(3o4) \ll I = \{0,1,2\} \ll I = \{0,1,2,3\}$. Thus, $I$ is a Hyper d$^*$-Ideal of $H$.

## Discussion

The introduction of hyper BCK-algebra (1999) lead to the development of different hyper algebraic structures. With the motivation of these results, we investigate the new concepts of Hyper d-Algebra, Hyper d-Ideals, Hyper d*-Algebra with different properties mentioned in the main result above. In general in this research work we used the methods of direct proof method, indirect proof and proof by contradiction methods.

## Conclusion

In this research paper the new concept of Hyper d-Algebra with different characterizations are discussed. This idea can be extended to other algebraic structures by including different properties with application**.**

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## Author’s Contributions

**Gerima Tefera:** Initializing the concept, methodology, written the manuscript, designed the study, checked the proof of theorems and propositions.

**Ali Mohammed:** Formalized the concepts, prove theorems, provide methods, checked concepts and verify with examples.

## Ethics

The authors mentioned in the paper read, approved and give consistent comment. We approved that it is not under review for publication and it is not accepted for publication in other journal.

### *Conflicts of Interest*

The authors declared that they have no conflicts of interest.

### *Data Availability*

The data used to support the findings of this study is indicated by citation within the study of the article.

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